# Optimizing pig marketing decisions under price fluctuations*

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16-06-2019

**Abstract:** In the manufacturing of fattening pigs, pig marketing refers to a sequence of culling decisions until the production unit is empty. The profit of a production unit is highly dependent on the price of pork, the cost of feeding and the cost of buying piglets. Price fluctuations in the market consequently influence the profit, and the optimal marketing decisions may change under different price conditions. Most studies have considered pig marketing under constant price conditions. However, because price fluctuations have an influence on profit and optimal marketing decisions it is relevant to consider pig marketing under price fluctuations. In this paper we formulate a hierarchical Markov decision process with two levels which model sequential marketing decisions under price fluctuations in a pig pen. The state of the system is based on information about pork, piglet and feed prices. Moreover, the information is updated using a Bayesian approach and embedded into the hierarchical Markov decision process. The optimal policy is analyzed under different patterns of price fluctuations. We also assess the value of including price information into the model.
*Keywords:* Markov decision process Bayesian updating pig production herd management price fluctuations.

## 1 Introduction

In the production of fattening pigs, one of the main managerial decisions is *pig marketing* (Kure, 1997). It refers to a sequence of culling decisions until the production unit is empty. The profit at marketing depends on endogenous factors such as growth, housing conditions and management policy as well as exogenous factors such as market prices. Prices of pork, piglets and feed may fluctuate in the market on a weekly basis and hence the farmer should take into account the influence of price fluctuations when choosing when to send animals to the abattoir.

In a production system of growing/finishing pigs, the animals are considered at different levels: herd, section, pen, or animal. The herd is a group of sections, a section includes some pens, and a pen involves some animals. In the production of fattening pigs, a farmer either buys piglets on the market or transfers them from another production unit when they weigh approximately 30 kilograms. The pigs are inserted into a *finisher pen* where they grow for 9 to 15 weeks until marketing. Because pigs in general grow at different rates, they obtain their slaughter weight at different times in the last weeks of the growing period. At the end of the growing period the farmer should therefore determine which pigs should be selected for slaughter (individual marketing). The reward of marketing a pig depends on the *pork price* of the carcass weight, the cost of buying the piglet on the market (the *piglet price*)

*Preprint of R. Pourmoayed and L.R. Nielsen, An approximate dynamic programming approach for sequential pig marketing decisions at herd level in Annals of Operations Research (2020), doi:10.1007/s10479-020-03646-0

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and the cost of feeding which is dependent on the *feed price* at the time when the feed stock is bought (e.g., at the start of the production cycle). Next, after a sequence of individual marketings, the farmer must decide when to *terminate* (empty) the rest of the pen. Terminating a pen means that the remaining pigs in the pen are sent to the abattoir (in one delivery) and after cleaning the pen, another group of piglets (each weighing approximately 30 kilograms) is inserted into the pen and *the production cycle* is repeated. That is, the farmer must time the marketing decisions while simultaneously considering the carcass weight, the length of the production cycle and exogenous price conditions. For an extended overview over pig production of growing pigs, see Pourmoayed and Nielsen (2014).

The problem of finding the optimal pig marketing policy, i.e., an optimal sequence of culling decisions until the production unit is empty, has been studied by a variety of researchers under different conditions. Chavas, Kliebenstein, and Crenshaw (1985) showed the importance of the animal growth on the marketing decisions by using the concepts of optimal control theory. Toft, Kristensen, and Jørgensen (2005) used a multi-level hierarchical Markov decision process to optimize the delivery strategy of pigs to the abattoir and to control epidemic diseases simultaneously. Boys, Li, Preckel, Schinckel, and Foster (2007) determined the optimal slaughter weight of pigs using a simulation approach to utilize the full capacity of trucks for delivering the pigs to the abattoir. In the study by Niemi (2006), a stochastic dynamic programming method was used to find the optimal marketing policy under the best nutrient ingredients of feed rations. Ohlmann and Jones (2011) considered the effect of stocking space and shipping on the problem and found the best timing of delivery to the packers using a mixed-integer linear programming model. Kristensen, Nielsen, and Nielsen (2012) proposed a two-level hierarchical Markov decision process and a state space model to optimize the marketing policy of the farm under online information acquired from sensor data. In the study by Plà, Rodriguez-Sanchez, and Rebillas-Loredo (2013), the optimal marketing policy was found by a mixed integer linear programming method under an all-in all-out strategy. Khamjan, Piewthongngam, and Pathumnakul (2013) considered a two-level supply chain of fattening units (as supplier) and abattoir (as buyer) to find the best procurement plan of buying the pigs from a zone of farms. They formulated the problem by a mathematical programming model and solved their model using a heuristic approach under different pig size distributions and pig growth rates.

The above mentioned studies investigated the marketing policy under constant price conditions. Only a few studies take price fluctuations into account. Broekmans (1992) analyzed the effect of price fluctuations on the marketing policy of fattening pigs using a first order autoregressive model proposed by Jørgensen (1992) with a limited range of possible price values. Moreover, learning aspects of price parameters from the historical data were not taken into account in this research. In the study by Roemen and de Klein (1999), only a fluctuating pork price was considered and the piglet price was modeled as a constant factor of the pork price. They used a Markov decision process to model the sequential marketing decisions under pork price fluctuations but no numerical example was given to show the efficiency of the proposed model.

In order to close this gap in the literature, we consider pig marketing at pen level under three price fluctuations, namely, the pork, piglet and feed price. The modeling process is formulated within a *business analytics* framework and the three categories descriptive, predictive and prescriptive analytics (Lustig, Dietrich, Johnson, and Dziekan, 2010). Descriptive analytics use data to get an insight of the past and identify possible predictive models (what has happened?). Predictive analytics use data and mathematical techniques to predict future outcomes (what could happen?). Prescriptive analytics consider decision models given objectives and constraints, with the goal of improving business performance (what should we do?). The contributions of the paper can be summarized as follows:

- A novel prescriptive model is formulated using a hierarchical Markov decision process (HMDP) with two levels. The model considers sequential marketing decisions at pen level and the state of the system is based on information about pork, piglet and feed prices.

- A set of predictive models are developed to model time series of pork, piglet and feed prices obtained from the market. Moreover, Bayesian forecasting is applied to update price information given the historical data. This is done using Gaussian state space models (SSMs, West and Harrison (1997)).
- The prescriptive model is formulated such that the predictive models are embedded into the HMDP. That is, the state of the system automatically adapts to new market prices (data) arriving and the optimal market decision is based on all previous historical data. The prescriptive model is a novel approach of taking fluctuating prices into account in the agribusiness.
- The prescriptive model is tested to obtain key findings of the optimal decisions under different scenarios of price fluctuations. Moreover, the value of including price information into the model is quantified.

The paper is organized as follows. First, in Section 2, we formulate the prescriptive model, i.e., the HMDP used for modeling sequential marketing decisions at pen level. Markov decision models are a well-known technique within animal science used to model livestock systems. See for instance Rodriguez, Jensen, Pla, and Kristensen (2011) and Nielsen, Jørgensen, Kristensen, and Østergaard (2010) and a recent survey by Nielsen and Kristensen (2014), which cites more than 100 papers using (hierarchical) Markov decision processes to model and optimize livestock systems. Section 3 presents the predictive models used by the prescriptive model, i.e., the SSMs for forecasting prices. An SSM consists of a set of latent variables and a set of observed variables which are linked together using the system and observation equations. At a specific point in time the estimated value of the latent variables may be considered as the state of the system, and using Bayesian forecasting we can estimate the state of the latent variables of the system when new data arrive. Examples of SSMs applied to agricultural problems are Cornou, Vinther, and Kristensen (2008) and Bono, Cornou, and Kristensen (2012); Bono, Cornou, Lundbye-Christensen, and Kristensen (2013). Section 3 also presents a procedure for embedding the models into the HMDP (Nielsen, Jørgensen, and Højsgaard, 2011). Next, in Section 4 we test the model under different scenarios and evaluate the value of including price information into the HMDP. Finally in Section 5, we conclude the paper and give directions for further research.

## 2 Prescriptive model

This study models pig marketing using a *hierarchical Markov decision process* (*HMDP*). A short introduction to HMDPs is given below. Because techniques from both statistical forecasting and operations research are used, a consistent notation can be hard to specify. To assist the reader, Appendix A provides an overview over the notation.

An HMDP is an extension of a *semi-Markov decision process* (*semi-MDP*) where a series of finite-horizon semi-MDPs are combined into one infinite time-horizon process at the founder level called the *founder process* (Kristensen and Jørgensen, 2000). The idea is to expand stages of a process to so-called *child processes*, which again may expand stages further to new child processes leading to multiple levels. At the lowest level, the HMDP consists of a set of finite-horizon semi-MDPs (see e.g., Tijms, 2003, Chap. 7). All processes are linked together using jump actions.

In order to have a frame of reference, we exploit the notation used for a semi-MDP and extend it to an HMDP. A finite-horizon semi-MDP models a sequential decision problem over $\mathscr{N}$ stages. Let

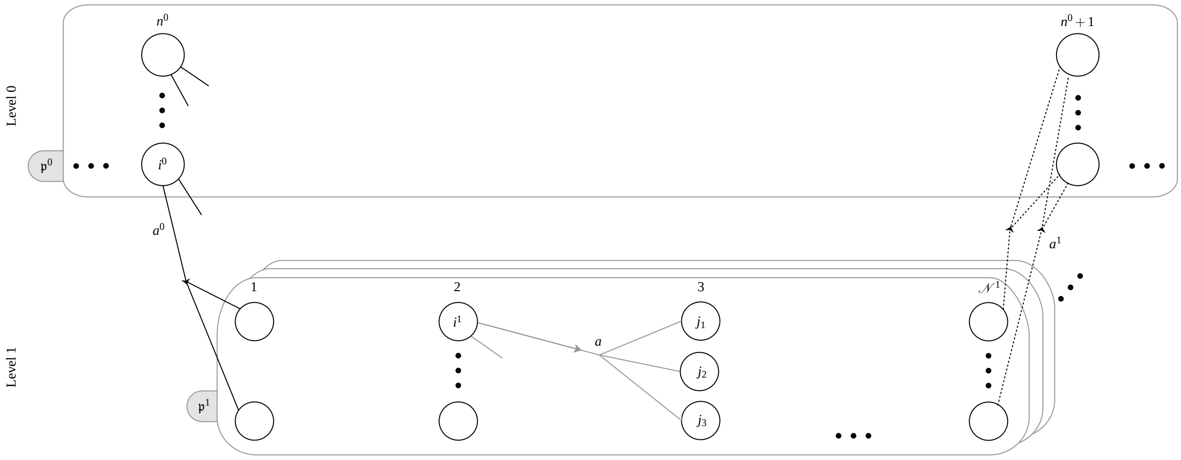


Figure 1: An illustration of a stage in an HMDP. At the founder level (Level 0) there is a single infinite-horizon founder process $\mathfrak{p}^0$. A child process, such as $\mathfrak{p}^1$ at Level 1 (oval box), is uniquely defined by a given stage, state (node), and action (hyperarc) of its parent process and linked with the parent process using its initial probability distribution (solid lines) and its terminating actions (dashed lines). Each process at level 2 is a semi-MDP. Note that only a subset of the actions are drawn.

$\mathbb{I}_n$ denote the finite set of system states at stage $n$. Given system state $i \in \mathbb{I}_n$ at stage $n$, an action $a$ from the finite set of allowable actions $\mathbb{A}_n(i)$ is chosen generating two outcomes: an immediate reward $r_n(i,a)$ and a probabilistic transition to state $j \in \mathbb{I}_{n+1}$ at stage $n+1$ with *transition probability* $\Pr(j \mid n,i,a)$. Moreover, let $u_n(i,a)$ denote the *stage length* of action $a$, i.e., the expected time until the next decision epoch (stage $n+1$) given action $a$ and state $i$.

An HMDP with two levels is illustrated in Figure 1 using a *state-expanded hypergraph* (Nielsen and Kristensen, 2006). At the first level, a single *founder process* $\mathfrak{p}^0$ is defined. Index 0 indicates that the process has no ancestral processes. Process $\mathfrak{p}^0$ is running over an infinite number of stages and all stages have identical state and action spaces and hence just a single stage is illustrated in Figure 1. Let $\mathfrak{p}^{l+1}$ denote a *child process* at level $l+1$. Process $\mathfrak{p}^{l+1}$ is uniquely defined by a given stage $n^l$, state $i^l$ and action $a^l$ of *parent process* $\mathfrak{p}^l$. For instance, the semi-MDP $\mathfrak{p}^1$ in Figure 1 is defined at stage $n^0$, state $i^0$ and action $a^0$ of the founder process $\mathfrak{p}^0$ symbolized by the notation $\mathfrak{p}^1 = (\mathfrak{p}^0 \parallel (n^0, i^0, a^0)\ )$. Each process is connected to its parent and child processes using *jump actions* which can be divided into two groups, namely, a *child jump action* that represents an *initial probability distribution* of transitions to a child process or a *parent jump action* that represents a *terminating probability distribution* of transitions to a parent process. This is illustrated in Figure 1 where child jump action $a^0$, illustrated using a solid *hyperarc*, represents a transition to the child process $\mathfrak{p}^1$ and parent jump action $a^1$ (illustrated using a dashed hyperarc) represents termination of the process $\mathfrak{p}^1$. Jump actions are like the traditional actions associated with an expected reward, action length, and a set of transition probabilities. Each node in Figure 1 at a given stage $n$ of a process $\mathfrak{p}^l$ corresponds to a state in $\mathbb{I}_n$. For example, there are three states at stage 3 in process $\mathfrak{p}^1$. Similarly, each gray hyperarc corresponds to an action, e.g., action $a$ results in a transition from state $i^1$ to either state $j_1, j_2$ or $j_3$.

A policy is a decision rule/function that assigns to each state in a process a (jump) action. That is, choosing a policy corresponds to choosing a single hyperarc out of each node in Figure 1. Given a policy, the reward at a stage of a parent process equals the total expected rewards of the corresponding child process. For instance, in Figure 1, the reward of choosing action $a^0$ in state $i^0$ at stage $n^0$ in process $\mathfrak{p}^0$ equals the total expected reward of process $\mathfrak{p}^1$. With a similar approach, the transition probabilities and the stage length of an action can be calculated at a stage of a parent process.

Different optimality criteria may be considered. In this paper, our optimality criterion is to maximize the *expected reward per time unit* and the optimal policy of the HMDP is found using a modified policy iteration algorithm. Due to the hierarchical structure, in general, the state space at the founder level can be reduced, and larger models can be solved (because the matrix which must be inverted in the modified policy iteration algorithm is much smaller). For a detailed description of the algorithm, the interested reader may consult Kristensen and Jørgensen (2000) and Nielsen and Kristensen (2014).

## 2.1 Assumptions

The HMDP for modeling marketing decisions in a finisher pen is formulated under the following assumptions:

1. the fixed number of pigs inserted into the pen at the beginning of each production cycle is $q^{\max}$;
2. marketing of pigs is started in week $t^{\min}$ at the earliest;
3. the pen is terminated in week $t^{\max}$ at the latest, i.e., the maximum life time of a pig in the pen is $t^{\max}$;
4. the sequence of feed-mixes used during the production cycle (feeding strategy) is known and fixed;
5. when a marketing decision happens, the preparation time for delivering the pigs to the abattoir is $b$ days;
6. weekly deliveries to the abattoir in the marketing period are based on a cooperative agreement where culled pigs from each pen are grouped into one delivery, i.e., the transportation cost is fixed;
7. marketing decisions are taken on a weekly basis, i.e., decision must be made $b$ days before each delivery;
8. after terminating the pen, the length of the period for cleaning the pen is $h$;
9. a new batch of piglets and the required feed stock are bought using market prices at the start of each production cycle;
10. the growth of a pig is independent of the other pigs in the pen, i.e., the growth does not depend on the number of pigs in the pen;
11. pigs are sold to the abattoir using the market pork price.

To give a complete description of the two-level HMDP with marketing decisions, the characteristics of each semi-MDP should be specified at all levels, i.e., stages, states, and (jump) actions including the corresponding rewards, stage lengths (measured in weeks), and transition probabilities.

## 2.2 Stages, states and actions

As illustrated in Figure 1, the founder process of the HMDP is an infinite time-horizon process where a stage represents a production cycle, i.e., the life of $q^{\max}$ pigs inserted into the pen (until termination). A stage of a process at the second level corresponds to either the period from insertion of the piglets until the marketing starts or a week in the marketing period (weeks $t^{\min}$ to $t^{\max}$). The length, stage, states, and (jump) actions of each process at levels 0 and 1 are described below. To avoid heavy notation, the superscript indicating the current level under consideration is left out whenever the level is clear from the context.

**Level 0 - Founder process $\mathfrak{p}^0$**

Stage: A production cycle of $q^{\max}$ pigs, i.e., from inserting the piglets into the pen until terminating the pen.

Time horizon: Infinite (the pen is filled and emptied an infinite number of times).

States: Due to the infinite time horizon, the state space is homogeneous and hence the stage index can be ignored when a state is defined at the founder process. A state $i^0 = \mathbb{p} \in \mathbb{P}$ represents our information about the pork, piglet and feeding prices (i.e., $\mathbb{I} = \mathbb{P}$). The price information is obtained from the market. Definition of $\mathbb{p}$ is given in Section 3.

Actions: For each state, a single child jump action $a^0$ (insertion of the piglets into the pen) is defined representing the initial probability distribution of transitions to the child process. The length of this action is zero.

Because the stage index can be ignored and there is only a single action, a child process is uniquely defined for each state $i^0 = \mathbb{p}$. That is, child process $\mathfrak{p}^1 = (\mathfrak{p}^0||n^0, i^0, a^0)$ is equivalent to $\mathfrak{p}^1 = (\mathfrak{p}^0||\mathbb{p})$.

**Level 1 - Child process $\mathfrak{p}^1 = (\mathfrak{p}^0||n^0, i^0, a^0)$**

Stage: The first stage ($n = 1$) represents the period from insertion of the piglets (week 1) until the start of marketing decisions (week $t^{\min}$). The remaining stages ($n > 1$) are one week in the marketing period (weeks $t^{\min}$ to $t^{\max}$). That is, stage $n = 1$ corresponds to the start of week 1 and stage $n > 1$ the start of week $n + t^{\min} - 2$.

Time horizon: Due to the definition of stages, the maximum number of stages is $\mathscr{N} = t^{\max} - t^{\min} + 2$.

States: Given stage $n$, a state $i$ is defined using state variables:

$\mathbb{d}_n$: information related to the deviations from the pork, piglet and feeding price information given in state $i^0$, obtained using Bayesian updating ($\mathbb{d}_n \in \mathbb{D}_n$). This information is obtained using the SSMs explained in Section 3.

$q_n$: number of pigs in the pen at the beginning of stage $n$.

Note that if $n \leq 2$ then $q_n = q^{\max}$. Hence the set of states becomes

$$\mathbb{I}_n = \left\{ i = (\mathbb{d}_n, q_n) \mid \mathbb{d}_n \in \mathbb{D}_n, q_n \in \{1 \cdot \mathbf{I}_{\{n>2\}} + q^{\max} \cdot \mathbf{I}_{\{n \leq 2\}}, \ldots, q^{\max}\} \right\},$$

where $\mathbf{I}_{\{\cdot\}}$ denotes the indicator function.

Actions: Consider state $i = (\mathbb{d}_n, q_n)$ at stage $n$. If $n = 1$, then marketing is not possible and the production process continues using action $a_{\texttt{cont}}$. If $1 < n < \mathscr{N}$ then the set of actions are $a_{\texttt{cont}}$, the parent jump action $a_{\texttt{term}}$ where the pen is terminated, and actions $a_q$, which implies that the $q$ heaviest pigs are culled (individual marketing). Finally, at the last stage $n = \mathscr{N}$, the pen must be terminated ($a_{\texttt{term}}$). Hence the set of actions becomes:

$$\mathbb{A}_n(i) = \begin{cases} \{a_{\texttt{cont}}\} & n = 1 \\ \{a_{\texttt{term}}, a_{\texttt{cont}}\} \cup \{a_q \mid 1 \leq q < q_n\} & 1 < n < \mathscr{N}, \\ \{a_{\texttt{term}}\} & n = \mathscr{N}. \end{cases} \quad (1)$$

The length of action $a_{\texttt{cont}}$ at stage 1 is $t^{\min} - 1$ weeks while for stage $n > 1$ the length of action $a_{\texttt{cont}}$ and $a_q$ is one week. For action $a_{\texttt{term}}$ the length is $h + b$ days.

## 2.3 Transition probabilities

**Founder process $\mathfrak{p}^0$**

Given stage $n^0$ and state $i^0 = \mathbb{p}$, a single child jump action $a^0$ was defined with a transition between the levels of the HMDP from state $i^0$ to state $j^1 = (\mathbb{d}_1, q_1)$ at the first stage of process $\mathfrak{p}^1$ ($n^1 = 1$). Because $q_1 = q^{\max}$, the transition probability becomes

$$\Pr\left(j^1 \mid n^0, i^0, a^0\right) = \Pr\left(\mathbb{d}_1 \mid \mathbb{p}\right), \tag{2}$$

where $\Pr\left(\mathbb{d}_1 \mid \mathbb{p}\right)$ is the initial probability of price deviations $\mathbb{d}_1$ given price information $\mathbb{p}$. The probability $\Pr\left(\mathbb{d}_1 \mid \mathbb{p}\right)$ depends on the statistical models used for Bayesian updating of the price information and will be explained in Section 3.

**Child process $\mathfrak{p}^1 = (\mathfrak{p}^0 || n^0, i^0, a^0)$**

As described in (1), for a given state $i = (\mathbb{d}_n, q_n)$ at stage $n$ of process $\mathfrak{p}^1$, there are three possible actions $a_{\mathtt{cont}}$, $a_q$ and $a_{\mathtt{term}}$.

Given actions $a_{\mathtt{cont}}$ or $a_q$, a transition occurs to state $j = (\mathbb{d}_{n+1}, q_{n+1})$ at the next stage of process $\mathfrak{p}^1$. If the process continues without marketing decisions, the only change of the system is related to the price deviation state variable. Hence, the transition probability is

$$\Pr\left(j \mid n, i, a_{\mathtt{cont}}\right) = \begin{cases} \Pr\left(\mathbb{d}_{n+1} \mid \mathbb{d}_n\right) & q_{n+1} = q_n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

If the $q$ heaviest pigs are culled from the pen, the transition probability becomes

$$\Pr\left(j \mid n, i, a_q\right) = \begin{cases} \Pr\left(\mathbb{d}_{n+1} \mid \mathbb{d}_n\right) & q_{n+1} = q_n - q, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

If the pen is terminated, a new production cycle is started with a transition to state $j^0 = \tilde{\mathbb{p}}$ at the next stage in process $\mathfrak{p}^0$. Hence, the transition probability becomes

$$\Pr\left(j^0 \mid n, i, a_{\mathtt{term}}\right) = \Pr\left(\tilde{\mathbb{p}} \mid \mathbb{d}_n\right), \tag{5}$$

where $\Pr\left(\tilde{\mathbb{p}} \mid \mathbb{d}_n\right)$ denotes the terminating probability of parent jump action $a_{\mathtt{term}}$.

Probabilities $\Pr\left(\mathbb{d}_{n+1} \mid \mathbb{d}_n\right)$ and $\Pr\left(\tilde{\mathbb{p}} \mid \mathbb{d}_n\right)$ depend on the statistical models used for Bayesian updating and will be described in Section 3.

## 2.4 Expected rewards

**Founder process $\mathfrak{p}^0$**

At the beginning of a production cycle, $q^{\max}$ piglets are inserted into the pen, i.e., the reward equals the cost of buying new piglets. That is, given state $i = \mathbb{p}$ and action $a^0$, the expected reward becomes

$$r_n(i, a^0) = -\mathbb{E}\left(p^{\mathtt{piglet}} q^{\max}\right), \tag{6}$$

where $p^{\mathtt{piglet}}$ is the price of one piglet at the beginning of the current production cycle.

**Child process $\mathfrak{p}^1 = (\mathfrak{p}^0||n^0, i^0, a^0)$**

At this level, the expected reward equals the expected revenue from selling the pigs minus the expected cost of feeding the remaining pigs conditioned on the values of the state variables and the action.

Consider state $i = (\mathbb{d}_n, q_n)$ at stage $n$ and let $(w_{(1)}, ..w_{(k)}, .., w_{(q_n)})_n$ denote the weight distribution of the pigs in the pen such that $w_{(1)}$, $w_{(k)}$, and $w_{(q_n)}$ are ordered random variables (order statistics) related to the weight of the lightest, $k$th and the heaviest pigs in the pen at stage $n$, respectively.

If the process continues without marketing decisions, the reward equals the expected feeding cost of $q_n$ pigs until next decision epoch

$$r_n(i, a_{\texttt{cont}}) = -\mathbb{E}\left(p^{\texttt{feed}} \sum_{k=1}^{q_n} f^{\texttt{feed}}_{(k),n}(t)\right), \tag{7}$$

where $p^{\texttt{feed}}$ is the feed price of one feed unit (FEsv[1]) at the beginning of the current production cycle and $f^{\texttt{feed}}_{(k),n}(t)$ denotes the expected feed intake of the $k$th lightest pig from the start of stage $n$ and the next $t$ days ahead. Note that when $n = 1$ and $n > 1$, $t$ will be equal to $7(t^{\min} - 1)$ and 7, respectively (see Section 2.2).

If the $q$ heaviest pigs are culled and the remaining $q_n - q$ pigs are kept in the pen, the expected reward of action $a_q$ becomes

$$\begin{aligned} r_n(i, a_q) = \mathbb{E}\left(\sum_{k=q_n-q+1}^{q_n} \tilde{w}_{(k)} \cdot p^{\texttt{pork}}_{(k),n}(\tilde{w}_{(k)}, \breve{w}_{(k)})\right) - \mathbb{E}\left(p^{\texttt{feed}} \sum_{k=q_n-q+1}^{q_n} f^{\texttt{feed}}_{(k),n}(b)\right) \\ - \mathbb{E}\left(p^{\texttt{feed}} \sum_{k=1}^{q_n-q} f^{\texttt{feed}}_{(k),n}(7)\right), \end{aligned} \tag{8}$$

where $\tilde{w}_{(k)}$ and $\breve{w}_{(k)}$ denote the carcass weight (kilograms) and the leanness (non-fat percentage) of the $k$th lightest pig in the pen at delivery, respectively. The price function $p^{\texttt{pork}}_{(k),n}(\cdot)$ is the *settlement pork price* of one kilogram of meat at delivery to the abattoir. This price may be different than the market pork price which is the price given if the pigs are in perfect condition. In (8) the first term is the reward of culling the pigs, the second term is the feeding cost of the culled pigs until they are sent to the abattoir, and the last term is the feeding cost of the remaining pigs.

Finally, if the pen is terminated, the expected reward becomes

$$r_n(i, a_{\texttt{term}}) = \mathbb{E}\left(\sum_{k=1}^{q_n} \tilde{w}_{(k)} \cdot p^{\texttt{pork}}_{(k),n}(\tilde{w}_{(k)}, \breve{w}_{(k)})\right) - \mathbb{E}\left(p^{\texttt{feed}} \sum_{k=1}^{q_n} f^{\texttt{feed}}_{(k),n}(b)\right). \tag{9}$$

To calculate the expected values in equations (6) to (9), more information is needed: the order statistics of the weights in the pen; transformation of weight to carcass weight and leanness; the feed intake, and the settlement pork price functions; the pork, feed and piglet prices. A random regression model is used to estimate the mean weight and standard deviation in the pen at a given week (Cai, Wu, and Dekkers, 2011) and hence the distribution of the ordered weights can be calculated (Pitmand, 1993). Formulas for the carcass weight and leanness of a pig given its weight are given in Appendix B.2. The feed intake function is based on biological relationships between weight, growth and feed intake, while the settlement pork price function is a piecewise linear function depending on the carcass weight, leanness and the market pork price. Due to the limited space, further details are given in

[1] Danish pig feed unit (1 FEsv = 7.72 MJ)

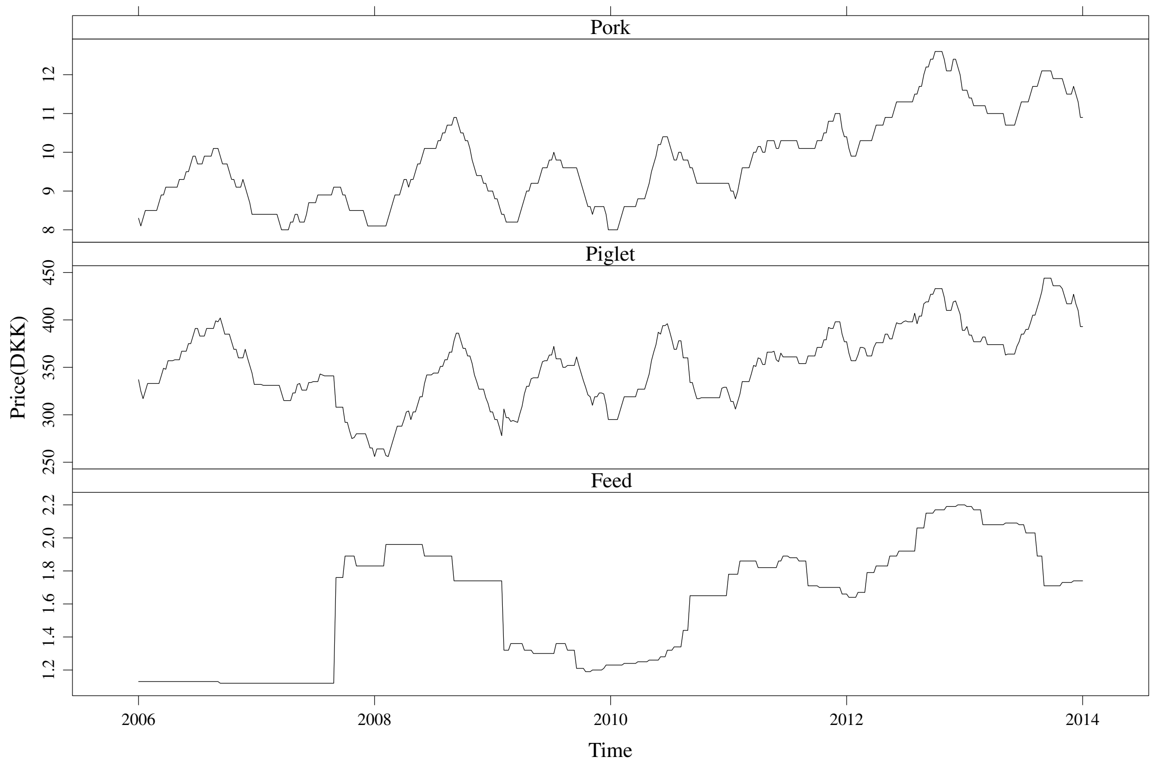


Figure 2: Weekly price data of pork, piglet and feed prices in Denmark (years 2006-2014[2]) in DKK. The pork price is the price of the carcass at the abattoir per kilogram when the total carcass weight is between 70 and 95 kilograms. The piglet price is the price of one piglet with a weight of approximately 30 kilograms. The feed price is the price per feed unit (FEsv - equivalent to 7.72 MJ).

Appendix B. Finally, information about the pork, feed and piglet prices is embedded into the HMDP using state space models based on Bayesian updating. The state space models are described in the next section.

# 3 Predictive models

The revenue of the pigs in a production cycle depends on pork, piglet and feed prices which fluctuate on the market every week. Figure 2 shows weekly changes of these prices in Denmark in the period of 2006 to the end of 2014. To transform these price data into information to be embedded into the states in the HMDP, we need a statistical model for time-series analysis. In our case, due to the non-stationary behavior of price data, we use a *state space model* (*SSM*, West and Harrison (1997)). An SSM consists of a set of latent variables and a set of observed variables. At a specified point in time the conditional distribution of the observed variables is a function of the latent variables specified via the observation equations. The latent variables change over time as described via the system equations. The observations are conditionally independent given the latent variables. Thus the estimated value of

[2] Time series of pig, piglet and feed prices in Denmark can be found on `http://www.notering.dk/WebFrontend/`. Pork and feed prices are for finisher pigs, and the piglet price is the "30 kilogram basic" price.

the latent variables at a time point may be considered as the state of the system, and Bayesian updating (the Kalman filter) can be applied to estimate the latent variables (real state) of the system via the observed variables. SSMs can be categorized into different groups based on the dynamic nature of the system considered and the probability distribution assumed. In this paper, the probability distribution of the prices is Gaussian and the dynamics of the system are modeled by linear equations. For a short introduction to SSMs and the theory used for Bayesian updating, see Appendix C.

### 3.1 SSMs for price prediction

We formulate three SSMs for the pork, piglet and feed prices to be embedded into the HMDP in Section 3.2.

**Pork price**

In order to estimate weekly price deviations and to forecast future pork prices, a *local linear trend* SSM (Durbin and Koopman, 2012, page 44) is used:

$$\begin{aligned}
&\text{Observation equation } (y_t = F'\theta_t + v_t): && p_t^{\texttt{pork}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_t^{\texttt{pork}} \\ \lambda_t^{\texttt{pork}} \end{pmatrix}, \\
&\text{System equation } (\theta_t = G\theta_{t-1} + \omega_t): && \begin{pmatrix} \mu_t^{\texttt{pork}} \\ \lambda_t^{\texttt{pork}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1}^{\texttt{pork}} \\ \lambda_{t-1}^{\texttt{pork}} \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_t^{\texttt{pork}} \end{pmatrix},
\end{aligned} \tag{10}$$

where $p_t^{\texttt{pork}}$ is the observed pork price at time $t$, the latent variable $\lambda_t^{\texttt{pork}}$ is the slope parameter that represents the deviation of pork price at time $t$ from the price at time $t-1$, $\mu_t^{\texttt{pork}} = p_t^{\texttt{pork}}$ is a supplementary latent variable, and $\omega_t^{\texttt{pork}} \sim N(0, W^{\texttt{pork}})$ is a random term. The initial prior distribution is $\theta_0 \sim N(m_0^{\texttt{pork}}, C_0^{\texttt{pork}})$.

**Feed price**

A *local level* SSM (Durbin and Koopman, 2012, Page 9) is used to model the feed price:

$$\begin{aligned}
&\text{Observation equation } (y_t = F'\theta_t + v_t): && p_t^{\texttt{feed}} - p^{\texttt{feed}} = \lambda_t^{\texttt{feed}} + v_t^{\texttt{feed}}, \\
&\text{System equation } (\theta_t = G\theta_{t-1} + \omega_t): && \lambda_t^{\texttt{feed}} = \lambda_{t-1}^{\texttt{feed}} + \omega_t^{\texttt{feed}},
\end{aligned} \tag{11}$$

where the observed variable $y_t = p_t^{\texttt{feed}} - p^{\texttt{feed}}$ at time $t$ denotes the difference between the current feed price $p_t^{\texttt{feed}}$ and the observed feed price $p^{\texttt{feed}}$ at the start of the current production cycle. The latent variable $\lambda_t^{\texttt{feed}}$ shows the deviation of feed price from $p^{\texttt{feed}}$. $\omega_t^{\texttt{feed}} \sim N(0, W^{\texttt{feed}})$ and $v_t^{\texttt{feed}} \sim N(0, V^{\texttt{feed}})$ are two random terms. The initial prior distribution is $\theta_0 \sim N(m_0^{\texttt{feed}}, C_0^{\texttt{feed}})$.

**Piglet price**

According to Figure 2, when the pork price is high, the piglet price $p_t^{\texttt{piglet}}$ is also high and generally they follow each other (e.g., during year 2014 their correlation is 93%). This is a known relation, see e.g., Roemen and de Klein (1999) and Broekmans (1992). Hence, the fraction $p_t^{\texttt{piglet}}/p_t^{\texttt{pork}}$ is approximately constant given $t$. Therefore, the piglet price can be estimated given the pork price.

However, to increase our precision, we may apply a logarithmic transformation and update the deviation between the logarithms of piglet and pork prices using a local level SSM:

$$
\begin{aligned}
\text{Observation equation } (y_t = F_t'\theta_t + \nu_t): \quad d_t^{\texttt{piglet}} &= \lambda_t^{\texttt{piglet}} + \nu_t^{\texttt{piglet}}, \\
\text{System equation } (\theta_t = G_t\theta_{t-1} + \omega_t): \quad \lambda_t^{\texttt{piglet}} &= \lambda_{t-1}^{\texttt{piglet}} + \omega_t^{\texttt{piglet}},
\end{aligned}
\tag{12}
$$

where $d_t^{\texttt{piglet}} = log(p_t^{\texttt{piglet}}) - log(p_t^{\texttt{pork}})$ is the *log transformed observed piglet ratio*, and $\lambda_t^{\texttt{piglet}}$ is a latent variable for the deviation between these logarithms. $\omega_t^{\texttt{piglet}} \sim N(0, W^{\texttt{piglet}})$ and $\nu_t^{\texttt{piglet}} \sim N(0, V^{\texttt{piglet}})$ are two error terms. The initial prior is $\theta_0 \sim N(m_0^{\texttt{piglet}}, C_0^{\texttt{piglet}})$.

## 3.2 Embedding the SSMs into the HMDP

The three SSMs described in the previous section provide information about the current prices. Given one of the SSMs, let $\mathbb{D}_{t-1} = (y_1, ..., y_{t-1}, m_0, C_0)$ denote the information available up to time $t-1$. Each time new data arrive, Bayesian updating (Theorem 1 in Appendix C) can be used to update the posterior $(\theta_t \mid \mathbb{D}_t) \sim N(m_t, C_t)$ at time $t$. That is, $m_t$ is our best estimate of the latent variable or price information.

Hence, to embed this information into the HMDP, the state variables $\mathbb{p}$ and $\mathbb{d}_n$ are defined to represent price information at Levels 0 and 1 as

$$
\mathbb{p} = \left(p^{\texttt{pork}}, p^{\texttt{feed}}, d^{\texttt{piglet}}\right), \tag{13}
$$

$$
\mathbb{d}_n = \left(m_n^{\texttt{pork}}, m_n^{\texttt{feed}}, m_n^{\texttt{piglet}}\right) = \left((\hat{\mu}_n^{\texttt{pork}}, \hat{\lambda}_n^{\texttt{pork}}), \hat{\lambda}_n^{\texttt{feed}}, \hat{\lambda}_n^{\texttt{piglet}}\right), \tag{14}
$$

where $p^{\texttt{pork}}$ and $p^{\texttt{feed}}$ denote the observed pork and feed prices at the start of a production cycle and $d^{\texttt{piglet}}$ is the log transformed observed piglet ratio. Similarly, $(\hat{\mu}_n^{\texttt{pork}}, \hat{\lambda}_n^{\texttt{pork}})$, $\hat{\lambda}_n^{\texttt{feed}}$ and $\hat{\lambda}_n^{\texttt{piglet}}$ denote the posterior mean values of the latent variables in the SSMs for pork, feed and piglet prices, respectively.

States $\mathbb{p}$ and $\mathbb{d}_n$ are used to calculate our expected rewards in Section 2.4. The piglet price used in (6) is $p^{\texttt{piglet}} = p^{\texttt{pork}} \exp(d^{\texttt{piglet}})$, the feed price used in (7) is $p^{\texttt{feed}}$, and the settlement pork price function in (21), when pigs are culled at level 1, uses the market pork price $\hat{\mu}_n^{\texttt{pork}}$.

Moreover, states $\mathbb{p}$ and $\mathbb{d}_n$ are used to calculate transition probabilities. Before calculating the transition probabilities, a discretization approach should be specified for the continuous state variables in (13) and (14) because the state space in the HMDP is discrete (Nielsen et al., 2011). Let $\mathbb{U}_{x_n} = \{\Pi_1, \ldots, \Pi_{|\mathbb{U}_{x_n}|}\}$ be a set of disjoint intervals representing the partitioning of possible values of the continuous state variable $x_n$ at stage $n$ (e.g., $x_n = \hat{\mu}_n^{\texttt{pork}}$). Moreover, given interval $\Pi$, let *center point* $\pi$ denote the center of the interval. That is, a possible value of state variable $x_n$ can be represented by center point $\pi_{x_n}$ in interval $\Pi_{x_n}$. As a result the state sets at Levels 0 and 1 (see Section 2.2) become

$$
\begin{aligned}
\mathbb{P} &= \mathbb{U}_{p^{\texttt{pork}}} \times \mathbb{U}_{p^{\texttt{feed}}} \times \mathbb{U}_{d^{\texttt{piglet}}}, \\
\mathbb{D}_n &= \mathbb{U}_{\hat{\mu}_n^{\texttt{pork}}} \times \mathbb{U}_{\hat{\lambda}_n^{\texttt{pork}}} \times \mathbb{U}_{\hat{\lambda}_n^{\texttt{feed}}} \times \mathbb{U}_{\hat{\lambda}_n^{\texttt{piglet}}}.
\end{aligned}
$$

Now the transition probabilities (2)-(5) in Section 2.3 can be calculated. First, consider child jump probability (2) with a transition to stage 1 at Level 1. This transition is deterministic

$$
\Pr(\mathbb{d}_1 \mid \mathbb{p}) = \begin{cases} 1 & \mathbb{d}_1 = ((p^{\texttt{pork}}, 0), 0, d^{\texttt{piglet}}), \\ 0 & \text{otherwise}, \end{cases}
$$

because $\hat{\mu}_1^{\texttt{pork}} = p^{\texttt{pork}}$ due to (10), $\hat{\lambda}_1^{\texttt{pork}}$ is assumed zero, $\hat{\lambda}_1^{\texttt{feed}} = 0$ due to (11) and $\hat{\lambda}_1^{\texttt{piglet}} = d^{\texttt{piglet}}$ due to (12).

Next, consider the transition probability $\Pr(\mathbb{d}_{n+1} \mid \mathbb{d}_n)$ for the actions $a_{\texttt{cont}}$ and $a_q$ used in (3) and (4). Because $\mathbb{d}_n$ includes state variables related to the three independent SSMs of pork, feed, and piglet prices, this probability equals to

$$\begin{aligned}\Pr(\mathbb{d}_{n+1} \mid \mathbb{d}_n) =& \Pr\left(m_{n+1}^{\texttt{pork}} \mid m_n^{\texttt{pork}}\right) \cdot \Pr\left(m_{n+1}^{\texttt{feed}} \mid m_n^{\texttt{feed}}\right) \cdot \Pr\left(m_{n+1}^{\texttt{piglet}} \mid m_n^{\texttt{piglet}}\right) \\ =& \Pr\left((\mu_{n+1}^{\texttt{pork}}, \lambda_{n+1}^{\texttt{pork}}) \in \Pi_{\mu_{n+1}^{\texttt{pork}}} \times \Pi_{\lambda_{n+1}^{\texttt{pork}}} \mid (\pi_{\mu_n^{\texttt{pork}}}, \pi_{\lambda_n^{\texttt{pork}}})\right) \\ & \cdot \Pr\left(\lambda_{n+1}^{\texttt{feed}} \in \Pi_{\lambda_{n+1}^{\texttt{feed}}} \mid \pi_{\lambda_n^{\texttt{feed}}}\right) \cdot \Pr\left(\lambda_{n+1}^{\texttt{piglet}} \in \Pi_{\lambda_{n+1}^{\texttt{piglet}}} \mid \pi_{\lambda_n^{\texttt{piglet}}}\right).\end{aligned}$$

Notice that due to our discretization approach, the probabilities are calculated over intervals given previous center points. Moreover, the conditional probability distributions of $(m_{n+1}|m_n)$ can be obtained using the $k$-step distribution defined in Theorem 2 in Appendix C where $k$ denotes the length of the current stage.

Finally, for parent jump probability (5) to state $\tilde{\mathbb{p}} = (p^{\texttt{pork}}, p^{\texttt{feed}}, d^{\texttt{piglet}})$ used under action $a_{\texttt{term}}$, the probability becomes

$$\begin{aligned}\Pr(\tilde{\mathbb{p}} \mid \mathbb{d}_n) =& \Pr\left(p^{\texttt{pork}} \mid m_n^{\texttt{pork}}\right) \cdot \Pr\left(p^{\texttt{feed}} \mid m_n^{\texttt{feed}}\right) \cdot \Pr\left(d^{\texttt{piglet}} \mid m_n^{\texttt{piglet}}\right) \\ =& \Pr\left(p^{\texttt{pork}} \in \Pi_{p^{\texttt{pork}}} \mid (\pi_{\mu_n^{\texttt{pork}}}, \pi_{\lambda_n^{\texttt{pork}}})\right) \\ & \cdot \Pr\left(p^{\texttt{feed}} \in \Pi_{p^{\texttt{feed}}} \mid \pi_{\lambda_n^{\texttt{feed}}}\right) \cdot \Pr\left(d^{\texttt{piglet}} \in \Pi_{d^{\texttt{piglet}}} \mid \pi_{\lambda_n^{\texttt{piglet}}}\right).\end{aligned}$$

Conditional distributions $(\cdot|m_n)$, can be obtained using the $k$-step distribution defined in Theorem 2 in Appendix C where $k$ is the expected length of action $a_{\texttt{term}}$ ($k = h + b$ days).

# 4 Scenario evaluation and value of information

In this section, we calculate the optimal policy of the HMDP to investigate the influence of price fluctuations on the optimal marketing decisions. First, we consider three scenarios with different trends in the prices. This provides some insights how the optimal marketing decisions change based on the price information. Second, we consider the relevance of embedding a statistical model with fluctuating prices into the HMDP by comparing the expected reward per time unit in a model with and without price information, i.e., the value of price information.

## 4.1 Model parameters

In order to initialize the model, we need the parameter values of the HMDP and the statistical models embedded into the HMDP. The parameter values are given in Table 1. They have been obtained using historical pork, piglet and feed market prices, information about finisher pig production (Danish conditions) and related literature (see the footnotes in Table 1).

More precisely, the parameter values of the HMDP were set based on discussions with Danish experts in pig production, standard Danish herd conditions and related literature. The system and observational variances of each SSM modeling the pork, piglet and feed market prices were estimated using maximum likelihood estimation (MLE) applied to historical prices in Denmark from 2006-2014. To calculate the expected revenue of each state and action in the HMDP, we need to specify the settlement pork price $p^{\text{pork}}(\tilde{w}, \breve{w})$ which is a piecewise linear function under current Danish conditions and is specified in Appendix B. Moreover, to estimate parameters in the random regression model (RRM) for finding the weight distribution in the pen (see Appendix B), we used the restricted maximum likelihood method (RMLE) applied to a set of weight data acquired from a standard Danish herd. Finally, in order to formulate the HMDP, we need to specify possible values of the discrete state variable and the range of center points for the continuous state variables in the HMDP. Possible values for the discrete state variable are 1 to $q^{\max}$ (remaining pigs in the pen) and after model calibration, we divided the continuous state variables $p^{\texttt{pork}}$, $p^{\texttt{feed}}$, $d^{\texttt{piglet}}$, $\hat{\mu}_n^{\texttt{pork}}$, $\hat{\lambda}_n^{\texttt{pork}}$, $\hat{\lambda}_n^{\texttt{feed}}$, and $\hat{\lambda}_n^{\texttt{piglet}}$ into intervals with a given center point based on our discretization method in Section 3.2. An overview over the values of each state variable is given in Table 2.

## 4.2 Optimal marketing decisions under different scenarios

To analyze the behavior of the optimal policy under different patterns of price fluctuations we consider three scenarios, illustrated in Figure 3, over a period of 15 weeks assuming that the production cycle starts at week one and ends at the start of week 15 at the latest:

**Scenario 1:** Favorable trend of pork price and unfavorable trends of feed and piglet prices. Pork price increases from 10.3 to 11.3 DKK (9.7%), feed price increases from 1.79 to 1.92 DKK (7.3%) and piglet price increases from 336 to 396 DKK (17.9%). This scenario is based on the historical data from weeks 11-25 in 2012.

**Scenario 2:** Favorable trends of pork and feed prices and unfavorable trend of piglet price. Pork price increases from 10.3 to 11.3 DKK (9.7%), feed price decreases from 1.79 to 1.66 DKK (7.3%) and piglet price increases from 336 to 396 DKK (17.9%).

**Scenario 3:** Unfavorable trends of pork and feed prices and favorable trend of piglet price. Pork price decreases from 10.3 to 9.3 DKK (9.7%), feed price increases from 1.79 to 1.92 DKK DKK (7.3%) and piglet price decreases from 362 to 328 DKK (9.3%).

Note due to the high correlation between pork and piglet price data (Section 3.1) this is also assumed in the three scenarios.

During the 15 weeks period, the average weight in the pen increases from 26.8 to 128.9 kilograms with a standard deviation increasing from 3 to 15.4 kilograms (see Equation (16)). Notice that the growth of the pigs is the same in the three scenarios and hence the only factor affecting the marketing policy is the price information.

To find the optimal policy of the HMDP, the model was coded using the C++ programming language (gcc compiler) and R (R Core Team, 2015), and the optimal policy of the HMDP was calculated using the modified policy iteration algorithm[3] using the R package "MDP" (Nielsen, 2009). The source code is available on-line (Pourmoayed and Nielsen, 2015). Given the parameters in Table 1 and the discretization of state variables in Table 2, the number of states and actions at the founder level are both 1,200. Moreover, each child process at Level 1 contains 194,080 states and 1,412,080 actions. That is, the total numbers of states and actions of the model are 232,897,200 and 1,694,497,200, respectively. The optimal policy was found after approx. 10 hours.

[3]Using shared and external processes, i.e. the memory of child processes may be shared and loaded when needed. For further information see the documentation in Nielsen (2009).

Table 1: Parameter values.

| Parameter | Value | Description |
|---|---|---|
| <u>HMDP (Section 2)</u> | | |
| $q^{\max}$ | 15 | Number of pigs inserted into the pen.[a] |
| $t^{\max}$ | 14 | Maximum number of weeks in a production cycle.[a] |
| $t^{\min}$ | 9 | First possible week of marketing decisions.[a] |
| $h$ | 4 | Days used for cleaning the pen after termination.[a] |
| $b$ | 3 | Days before delivery to abattoir after a marketing decision.[a] |
| <u>SSMs (Section 3.1)</u> | | |
| $W^{\texttt{pork}}$ | $\begin{pmatrix}0 & 0\\ 0 & 0.173^2\end{pmatrix}$ | System variance (pork price).[b] |
| $W^{\texttt{feed}}$ | $0.044^2$ | System variance (feed price).[b] |
| $W^{\texttt{piglet}}$ | $0.0108^2$ | System variance (piglet price).[b] |
| $V^{\texttt{feed}}$ | 0 | Observation variance (feed price).[b] |
| $V^{\texttt{piglet}}$ | 0 | Observation variance (piglet price).[b] |
| $m_0^{\texttt{pork}}$ | $\begin{pmatrix}9.85 & 0\end{pmatrix}$ | Prior mean (pork price).[b] |
| $C_0^{\texttt{pork}}$ | $\begin{pmatrix}0 & 0\\ 0 & 0.139^2\end{pmatrix}$ | Prior variance (pork price).[b] |
| $m_0^{\texttt{feed}}$ | 0 | Prior mean (feed price).[b] |
| $C_0^{\texttt{feed}}$ | $0.336^2$ | Prior variance (feed price).[b] |
| $m_0^{\texttt{piglet}}$ | 3.55 | Prior mean (piglet price).[b] |
| $C_0^{\texttt{piglet}}$ | $0.057^2$ | Prior variance (piglet price).[b] |
| <u>Calculation of expected reward (Appendix B)</u> | | |
| $\beta$ | $\begin{pmatrix}21.767 & 4.914 & 0.149\end{pmatrix}'$ | Fixed parameters (RRM).[c] |
| $V$ | $\begin{pmatrix}2.072 & 0.828 & 0.01\\ 0.828 & 1.753 & -0.142\\ 0.01 & -0.142 & 0.015\end{pmatrix}$ | Covariance matrix for $\alpha_j$ (RRM).[c] |
| $R$ | 2.04 | Standard deviation of residual error (RRM).[c] |
| $\bar{g}$ | 6 | Average weekly gain (kilogram) in the herd.[d] |
| $\bar{\bar{w}}$ | 61 | Average leanness percentage in the herd.[d] |
| $\sigma_c^2$ | 1.4 | Standard deviation of conversion rate $c_s$.[d] |
| $k_2$ | 0.044 | Energy requirements (FEsv) per kilogram metabolic weight.[d] |
| $k_1$ | 1.549 | Energy requirement (FEsv) per kilogram gain.[d] |

[a] Value based on discussions with experts in Danish pig production. [b] Estimated based on time series of pig, feed and piglet prices that can be found on `http://www.notering.dk/WebFrontend/`. [c] Estimated using the weight data in a standard Danish herd. [d] Value taken from Kristensen et al. (2012).

Table 2: Cardinality of the discrete state variable and range of the center points for the continuous state variables.

| Process level | 0 | | | 1[a] | | | | |
|---|---|---|---|---|---|---|---|---|
| State variable | $p^{\texttt{pork}}$ | $p^{\texttt{feed}}$ | $d^{\texttt{piglet}}$ | $q_t$ | $\hat{\mu}_t^{\texttt{pork}}$ | $\hat{\lambda}_t^{\texttt{pork}}$ | $\hat{\lambda}_t^{\texttt{feed}}$ | $\hat{\lambda}_t^{\texttt{piglet}}$ |
| Intervals/cardinality | 16 | 15 | 5 | 15 | 16 | 5 | 5 | 5 |
| Range of center points | 9.2-12.2 | 1.5-2.2 | 3.4-3.6 | 1-15 | 9.2-12.2 | -0.4-0.4 | -0.1-0.1 | 3.5-3.7 |

[a] At stage 1 the only possible values are $q_t = 15$ and $\hat{\lambda}_t^{\texttt{pork}} = \hat{\lambda}_t^{\texttt{feed}} = 0$.

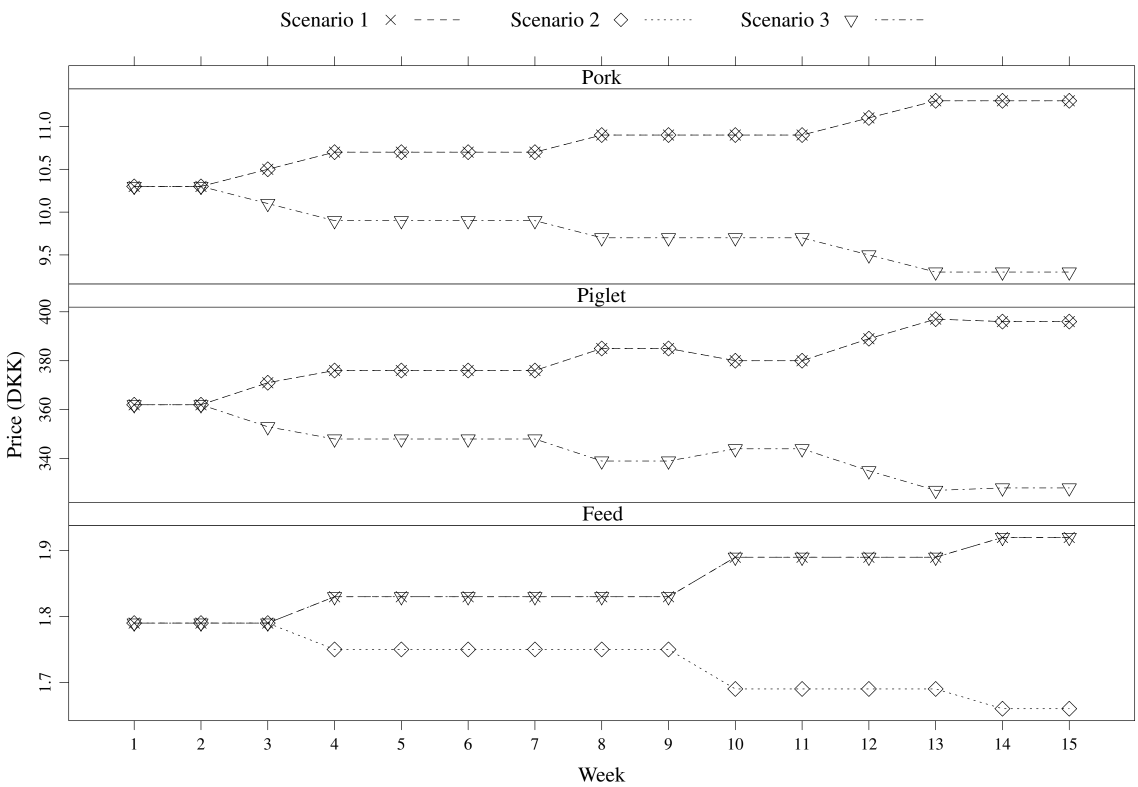


Figure 3: Price fluctuations in the three scenarios. In Scenario 1, the trends of feed and piglet prices are unfavorable and the trend of pork price is favorable. In Scenario 2, the trends of pork and feed prices are favorable and the trend of piglet price is unfavorable. In Scenario 3, the trends of pork and feed prices are unfavorable and the trend of piglet price is favorable.

For each scenario we use the SSMs to find the values of the state variables related to the price information in the HMDP. That is, for each scenario we identify the relevant state and the corresponding optimal action. The results for each scenario are illustrated in Figure 4 which include estimations of posterior mean parameters in the SSMs and the number of remaining pigs in the pen in each week (bars). The optimal decision $a^*$ is shown just above the x-axis where the numbers denote the number of the heaviest pigs culled from the pen ($a_q$), the letter "T" indicates the termination decision ($a_{\texttt{term}}$), and the letter "C" corresponds to continuing the production process without marketing decisions ($a_{\texttt{cont}}$).

In Scenarios 1 and 2, the trend of pork and piglet prices are the same (both increasing) while the trend in feed price are different. That is, by comparing the two scenarios, we observe the marginal

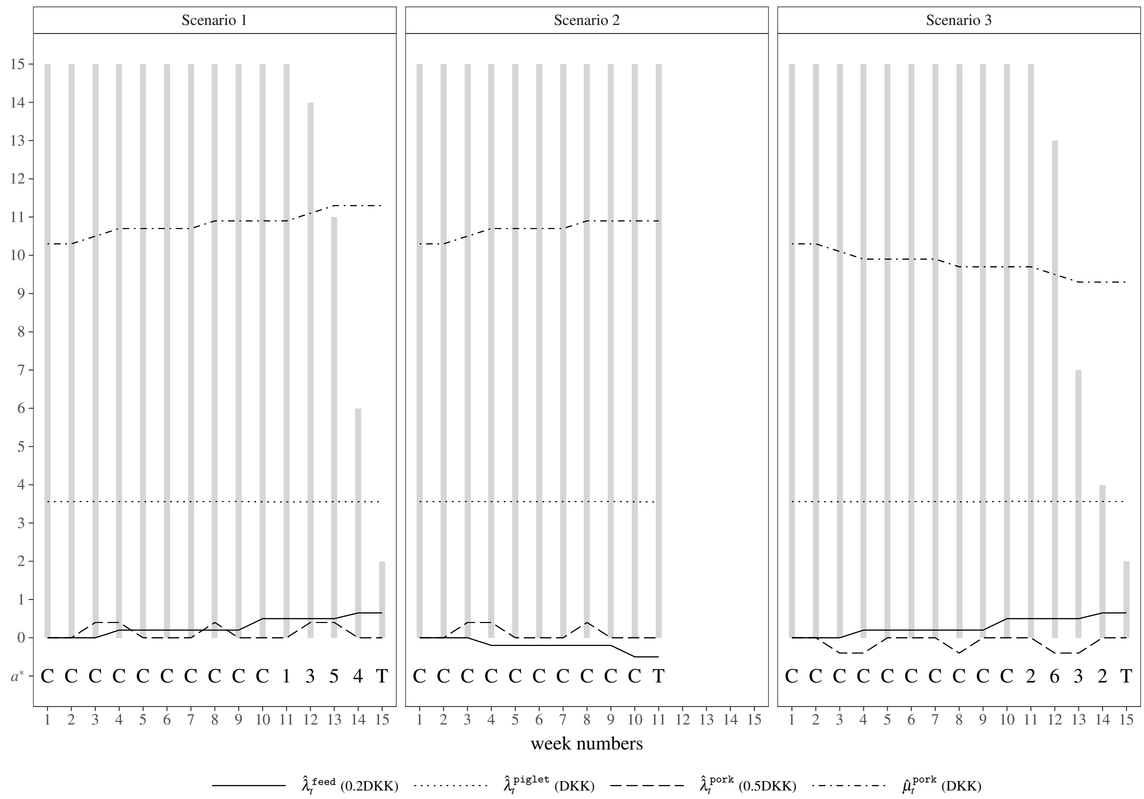


Figure 4: Estimated means of posterior parameters in the SSMs and the optimal decisions of the HMDP for the three scenarios. $\hat{\lambda}_t^{\texttt{pork}}, \hat{\lambda}_t^{\texttt{feed}}$, and $\hat{\lambda}_t^{\texttt{piglet}}$ are the mean estimates of pork, feed and piglet price deviations, respectively, and $\hat{\mu}_t^{\texttt{pork}}$ is the estimated mean of pork price. The optimal decision is shown just above the x-axis where the numbers denote the number of the heaviest pigs culled from the pen ($a_q$), the letter "T" indicates the termination decision ($a_{\texttt{term}}$), and the letter "C" corresponds to continuing the production process without marketing decisions ($a_{\texttt{cont}}$). The bars show the number of remaining pigs in the pen before making a decision. In the plot, the values of $\hat{\lambda}_t^{\texttt{pork}}$ and $\hat{\lambda}_t^{\texttt{feed}}$ have been scaled with factors 2 and 5, respectively.

effect of different trends of the feed price which have a significant impact on the optimal policy. In Scenario 2, a decreasing feed price leads to an earlier termination (week 11) compared to Scenario 1 with an increasing feed price (week 15). Note that due to Assumption 9 on page 5, when the pen is terminated, a low feed price affects the feeding cost of the next production cycle and hence when the feed price is low, it may be beneficial to terminate the pen earlier and start a new production cycle which happens in Scenario 2 where the feed price at termination (week 11) is 11.8% lower than the feed price in the same week in Scenario 1. On the other hand, an increasing feed price (in Scenario 1) during the marketing period (an increase from 1.83 to 1.92 in weeks 9-15) results in a longer production cycle and individual marketings in weeks 11 to 14.

In Scenario 3, we have an increasing trend in feed prices similar to Scenario 1 but unlike Scenario 1, the trends of pork and piglet prices are decreasing in this scenario (see Figure 3). That is, by comparing Scenarios 1 and 3 we observe the effect of different trends in the correlated pork and piglet prices (increasing and decreasing). Note marginally an increasing pork price results in a higher reward while an increasing piglet price results in a higher cost. Hence due to the conflicting factors the combined effect on profit is not as high as if only pork prices was increasing. Observing the optimal policies of Scenarios 1 and 3, increasing and decreasing pork and piglet prices do not change the week of termination (week 15). However, in Scenario 3 the fraction of remaining pigs in the pen in every week of the marketing period is lower than in Scenario 1. This is, the increasing trend of pork and piglet price in Scenario 1 that makes it more beneficial to keep more pigs in the pen and sell them in the following weeks while in Scenario 3 it is better to sell the pigs earlier since the pork price decreases in the following weeks.

By comparing equal trends of pork and piglet price (Scenarios 1 and 2) against equal trends of feed price (Scenarios 1 and 3), we see that the decreasing feed price in Scenario 2 has a higher impact on termination compared to changes in pork and piglet price. This observation was also given in Pourmoayed, Nielsen, and Kristensen (2016).

### 4.3 Value of price information

To determine the utility of embedding a statistical price model into the HMDP (a predictive price model), we compare the optimal policy of our model against the optimal policy of different models:

**Model I** Future prices are known. That is, given a price sample path the decision maker optimizes decisions based on full information about future prices (a model with full price information). If we simulate sample paths, the average reward per time unit obtained represent an upper bound.

**Model II** A rolling horizon approach where future prices are assumed the same as in the current epoch. That is, a deterministic model that reoptimizes each time prices change (a reactive model).

Under the expected reward per time unit criterion the difference between Model I and the HMDP equals the gain of using full price information compared to a predictive price model. Moreover, the difference between Model II and the HMDP equals the gain of using a reactive model compared to a predictive price model. That is, the difference is the extra reward per time unit gained by using Models I or II compared to the HMDP which embeds the SSMs for predicting the future market prices.

A descriptive summary is given in Table 3. The results were obtained using 1000 sample paths. A plot of the distributions of the differences are given in Figure 5. If full information is available (Model I) and the decision maker optimize decisions based on this, we on average obtain an extra reward of 30.31 DKK/week. Note Model I is not possible in practice, as it corresponds to having an oracle giving you future prices. However, it gives an upper bound on the possible profit. If a reactive model is considered (Model II), i.e., a rolling horizon approach where each week a new price is observed and the decision maker optimizes assuming this price in the future. Then, we on average obtain an extra reward of -34.49 DKK/week. Hence, the value of modeling price uncertainty using a predictive model is beneficial compared to a rolling horizon model. Finally, observe that the reward of the HMDP lies approximately in the middle between the two models indicating that a stochastic model gives better results than the reactive deterministic rolling horizon model, but obviously full information will give better results.

## 5 Discussion and Conclusions

In the production of fattening pigs, price fluctuations in the market have an effect on marketing decisions. In this paper we used a two-level HMDP to model marketing decisions under fluctuating pork,

Table 3: Descriptive summary.

| Model | $g^a$ | Difference | % |
|---|---|---|---|
| HMDP[b] | 129.99 | 0 | 0 |
| Model I | 160.30 | -30.31[c] | -18.91 |
| Model II | 95.05 | 34.49[d] | 36.76 |

[a] Average reward per time unit. [b] Base model comparing against. [c] Calculated using 1000 sample paths. Std. dev. = 81.52, 25% quantile = -88.18, 75% quantile = 28.63. [d] Calculated using 1000 sample paths. Std. dev. = 77.34, 25% quantile = -20.96, 75% quantile = 88.11.

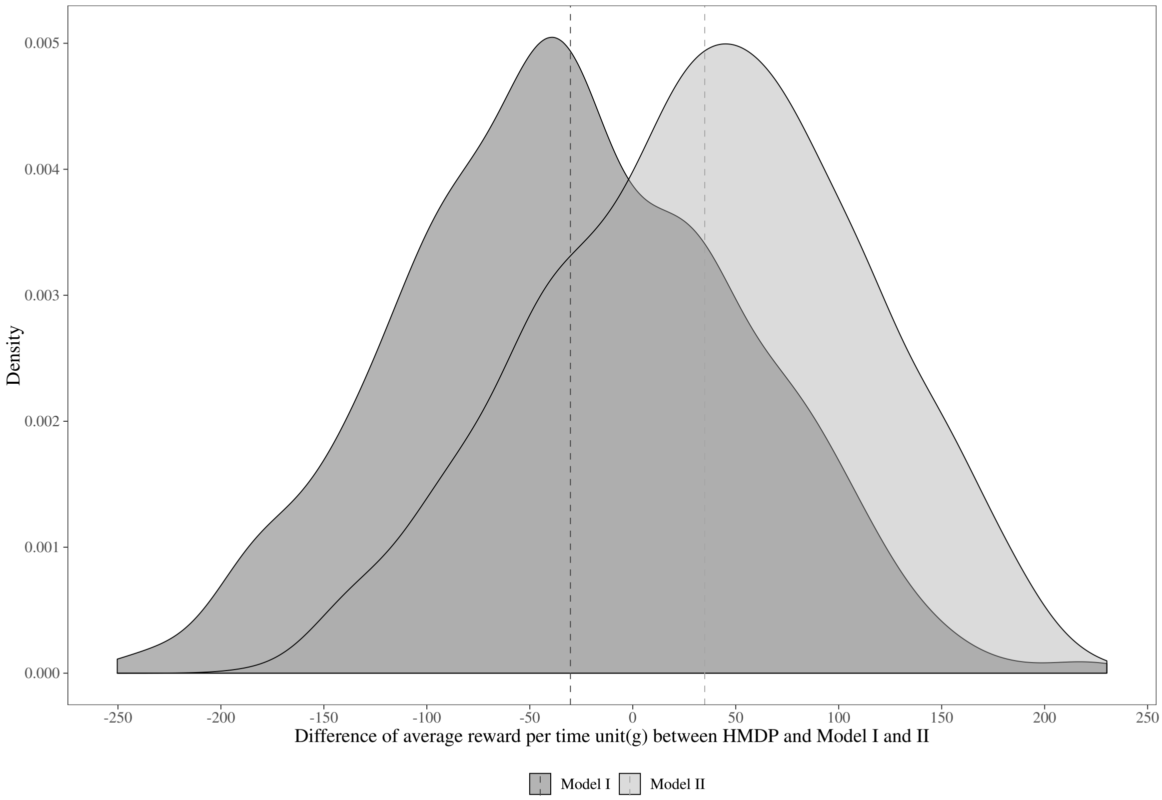


Figure 5: Density distribution of the difference between Model I (Model II) and the HMDP. Given the average reward per time unit $g_{\mathrm{H}}$ of the HMDP, the differences are calculated as $g_{\mathrm{H}} - g_{\mathrm{I}}^i$ and $g_{\mathrm{H}} - g_{\mathrm{II}}^i$ where $g_{\mathrm{I}}^i$ and $g_{\mathrm{II}}^i$ denote the reward of sample path $i$ of Model I and II, respectively. The vertical dashed lines are the means.

piglet and feed prices. Given a business analytics framework we formulate a novel prescriptive model using a hierarchical Markov decision process (HMDP) with two levels. The prescriptive model is a novel approach of taking fluctuating prices into account in the agribusiness.

We used a Bayesian approach to update the state of the system such that it contains updated information based on previous market prices. That is, three predictive models were formulated to forecast future prices based on historical data and each was embedded into the HMDP.

Key findings from the numerical experiments show that under the current model and parameters the optimal policy may be quite different given different price fluctuations; the effect of a fluctuating feed price was especially noticeable. Models assuming fixed prices do not take this into account. Finally, we analyzed the value of including information about fluctuating prices into the HMDP compared to using two different models providing and lower and upper bound on the reward. The results showed that the long-term average reward per time unit of the production unit can be improved by including price fluctuations into the model. That is, using data and analytics may be a good idea.

## Acknowledgements

This article has been written with support from The Danish Council for Strategic Research (The PigIT project, Grant number 11-116191). We would like to thank the two anonymous referees for improving this paper.

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# A Notation

Because the paper uses techniques from both statistical forecasting and operations research, we have to make some choices with respect to notation. In general, we use capital letters for matrices and let $A'$ denote the transpose of $A$. Capital blackboard bold letters are used for sets (e.g., $\mathbb{P}$ and $\mathbb{D}_n$). Finally, accent $\hat{x}$ (hat) is used to denote an estimate of $x$. A description of the notation introduced in Section 2 and Section 3 is given in Tables 4 and 5, respectively.

Table 4: Notation - HMDP model (Section 2).

| Symbol | Description |
|---|---|
| $\mathbb{I}_n$ | Set of states at stage $n$. |
| $\mathbb{A}_n(i)$ | Set of actions given stage $n$ and state $i$. |
| $r_n(i,a)$ | Reward at stage $n$ given state $i$ and action $a$. |
| $u_n(i,a)$ | Expected length until the next decision epoch at stage $n$ given state $i$ and action $a$. |
| $\Pr(j \mid n,i,a)$ | Transition probability from state $i$ at stage $n$ to state $j$ at the next stage under action $a$. |
| $\mathfrak{p}^l$ | A process at level $l$ (superscript is used to indicate level). |
| $\mathscr{N}^l$ | Time horizon of process $\mathfrak{p}^l$ at level $l$. |
| $n^l, i^l, a^l$ | A stage, state, and action in process $\mathfrak{p}^l$ . |
| $q^{\max}$ | Number of pigs inserted into the pen. |
| $t^{\max}$ | Latest possible week of pen termination. |
| $t^{\min}$ | First possible week of marketing decisions. |
| $h$ | Number of days for cleaning the pen after termination. |
| $b$ | Number of days of preparation for delivery to the abattoir. |
| $q_n$ | Remaining pigs in the pen at stage $n$, $1 \leq q_{n \leq} q^{\max}$. |
| $\mathbb{p}$ | Model information related to the price information in the first level of HMDP, $\mathbb{p} \in \mathbb{P}$. |
| $\mathbb{d}_n$ | Model information related to the price deviations in the second level of HMDP, $\mathbb{d}_n \in \mathbb{D}_n$. |
| $a_{\texttt{term}}$ | Action related to pen termination. |
| $a_{\texttt{cont}}$ | Action related to continuing the production process without marketing. |
| $a_q$ | Action related to marketing the $q$ heaviest pigs in the pen ($1 \leq q < q_n$). |
| $p^{\texttt{feed}}$ | Market feed price at the beginning of a production cycle (DKK). |
| $p^{\texttt{piglet}}$ | Market piglet price at the beginning of a production cycle (DKK). |
| $w_{(k)}$ | Weight of the $k$th pig in the pen (kilogram). |
| $f^{\texttt{feed}}_{(k),n}(t)$ | Expected feed intake of the $k$th lightest pig from the start of stage $n$ and the next $t$ days ahead (FEsv). |
| $\tilde{w}_{(k)}$ | Carcass weight of the $k$th lightest pig at delivery to the abattoir (kilogram). |
| $\check{w}_{(k)}$ | Leanness (non-fat percentage) of the $k$th lightest pig at delivery to the abattoir. |
| $p^{\texttt{pork}}_{(k),n}(\cdot)$ | Settlement pork price of the $k$th lightest pig of one kilogram of meat at delivery to the abattoir. |

Table 5: Notation - State space models (Section 3).

| Symbol | Description |
|---|---|
| $\theta_t$ | Latent/unobservable variable(s). |
| $y_t$ | Observable variable(s). |
| $G_t$ | Design matrix of system equation. |
| $F_t$ | Design matrix of observation equation. |
| $\omega_t$ | System noise, $\omega_t \sim N(0,W_t)$ where $W_t$ denotes the system covariance matrix. |
| $\nu_t$ | Observation error, $\nu_t \sim N(0,V_t)$ where $V_t$ denotes the observation covariance matrix. |
| $\mathbb{D}_t$ | Set of information available up to time $t$ in the system. |
| $(m_0,C_0)$ | Mean and covariance matrix of the prior, $\theta_0 \sim N(m_0,\, C_0)$. |
| $(m_t,C_t)$ | Mean and covariance matrix of the posterior at time $t$, $(\theta_t \vert \mathbb{D}_t) \sim N(m_t,\, C_t)$. |
| $p^{\texttt{pork}}_t$ | Observed market pork price at time $t$ (DKK). |
| $\mu^{\texttt{pork}}_t$ | A supplementary latent variable in the SSM of pork price ($\mu^{\texttt{pork}}_t = p^{\texttt{pork}}_t$ ). |
| $\lambda^{\texttt{pork}}_t$ | Price deviation related to pork price at time $t$. |
| $p^{\texttt{feed}}_t$ | Observed market feed price at time $t$ (DKK). |
| $\lambda^{\texttt{feed}}_t$ | Price deviation related to feed price at time $t$. |
| $p^{\texttt{piglet}}_t$ | Observed market piglet price at time $t$ (DKK). |
| $d^{\texttt{piglet}}_t$ | Log transformed observed piglet ratio. |
| $\lambda^{\texttt{piglet}}_t$ | Price deviation related to piglet price at time $t$. |
| $\mathbb{U}_{x_n}$ | Set of disjoint intervals representing the partitioning of state variable $x_n$ at stage $n$. |
| $\Pi_k$ | Interval $k$ in $\mathbb{U}_{x_n} = \{\Pi_1,..,\Pi_k,..,\Pi_{\vert \mathbb{U}_{x_n} \vert}\}$. |
| $\pi_k$ | Center point of interval $\Pi_k$. |

# B Calculating expected reward

## B.1 Modeling weights in the pen

During the growing period in the pen, pigs grow at different rates; that is, given a certain week in the production cycle, there are variation between the weights of the individual animals in the pen. Moreover, as the pigs grow, the variation increases and our uncertainty about the average weight of the pen increases.

Let $(w_{(1)},..w_{(k)},..,w_{(q)})_t$ denote the weight distribution of the $q$ pigs in the pen at week $t$ such that $w_{(1)}$, $w_{(k)}$, and $w_{(q)}$ are ordered random variables (order statistics) related to the weight of the lightest, $k$th lightest and the heaviest pig in the pen at stage $n$, respectively. To find the probability distribution of the ordered random variable $w_{(k)}$, first the weight distribution of a randomly selected pig should be determined in the pen. To specify this distribution, we use a *random regression model* (*RRM*), often applied in the animal breeding models (Schaeffer, 2004).

Let $w_{j,t}$ denote the weight of a randomly selected pig $j$ in week $t$ described using an RRM:

$$w_{j,t} = X_t\beta + Z_t\alpha_j + \varepsilon_{j,t}, \tag{15}$$

where $X_t$ and $Z_t$ are time covariate vectors, $\beta$ is the vector of fixed parameters, $\alpha_j$ is the vector of random parameters and $\varepsilon_{j,t}$ is a residual error. In this RRM, $X_t\beta$ is the fixed effect of the model representing the average weight of the pen and $Z_t\alpha_j$ is the random effect showing a deviation between the weight of pig $j$ and the average weight of the pen. A quadratic RRM is used suggested by Cai et al. (2011) where $X_t = Z_t = \begin{pmatrix}1 & t & t^2\end{pmatrix}$, $\alpha_j = \begin{pmatrix}\alpha_{0j} & \alpha_{1j} & \alpha_{2j}\end{pmatrix}'$ and $\beta = \begin{pmatrix}\beta_0 & \beta_1 & \beta_2\end{pmatrix}'$:

$$w_{j,t} = \beta_0 + \beta_1 t + \beta_2 t^2 + \alpha_{0j} + \alpha_{1j}t + \alpha_{2j}t^2 + \varepsilon_{j,t}.$$

Random parameter $\alpha_j$ follows a normal distribution with parameters

$$\alpha_j = \begin{pmatrix}\alpha_{0j}\\ \alpha_{1j}\\ \alpha_{2j}\end{pmatrix} \sim N(0, V = \begin{pmatrix}\sigma_0^2 & \sigma_{01} & \sigma_{02}\\ \sigma_{01} & \sigma_1^2 & \sigma_{12}\\ \sigma_{02} & \sigma_{12} & \sigma_2^2\end{pmatrix}),$$

where $V$ is independent of pig $j$ and time $t$. Moreover, the residual errors $\varepsilon_{j,t} \sim N(0,\, R)$ are independent random variables. Because (15) is linear with respect to random parameters $\alpha_j$ and $\varepsilon_{j,t}$, we can conclude

$$w_{j,t} \sim N\left(\mu_t = X_t\beta,\ \sigma_t^2 = Z_t V Z_t' + R)\right). \tag{16}$$

The parameters $\beta$, $V$ and $R$ can be estimated using the restricted maximum likelihood (REML) method (Patterson and Thompson, 1971).

Because the probability distribution of $w_{j,t}$ is independent of pig $j$, the weight distribution of all $q$ pigs in the pen are i.i.d at time $t$. Hence, the probability density function of the ordered random variable $w_{(k)}$ becomes

$$\phi_{(k)}(w) = \frac{q!}{(k-1)!(q-k)!}\Phi^{k-1}(w)[1-\Phi(w)]^{q-k}\phi(w),$$

where $\Phi(w)$ and $\phi(w)$ are the cumulative and density functions of the normal distribution defined in (16) (Pitmand, 1993, page 326).

## B.2 Carcass weight, leanness, feed intake and growth

Consider the $k$th lightest pig at stage $n$ with weight $w$ and daily growth $g$. The carcass weight $\tilde{w}$ can be approximated as

$$\tilde{w} = c_s w - 5.89 + e_{\mathtt{c}}, \tag{17}$$

where $e_{\mathtt{c}} \sim N(0, \sigma_c^2)$ (Andersen, Pedersen, and Ogannisian, 1999). The relation between growth rate, leanness (lean meat percentage) and feed conversion ratio varies widely between herds. Hence, these formulas must be herd specific. The leanness $\breve{w}$ can be found as

$$\breve{w} = \frac{-30(g - \bar{g})}{4} + \bar{\breve{w}}, \tag{18}$$

where $\bar{g}$ is the average daily growth in the herd and $\bar{\breve{w}}$ is the average herd leanness percentage (Kristensen et al., 2012).

The feed intake (energy intake) is modelled as the sum of feed for maintenance and feed for growth. The basic relation between daily feed intake $f$ (FEsv), live weight and daily gain is

$$f = k_1 g + k_2 w^{0.75}, \tag{19}$$

where $k_1$ and $k_2$ are constants describing the use of feed per kilogram gain and per kilogram metabolic weight, respectively (Jørgensen, 2003). As a result the expected feed intake of a pig over the next $\hat{t}$ days equals

$$f^{\mathtt{feed}}_{(k),n}(\hat{t}) = \mathbb{E}\left(\sum_{t=1}^{\hat{t}} f_t\right) = \mathbb{E}\left(\sum_{t=1}^{\hat{t}} \left(k_1 g + k_2 (w + (t-1)g)^{0.75}\right)\right) = \mathbb{E}\left(\hat{t} k_1 g + k_2 \sum_{t=1}^{\hat{t}} (w + (t-1)g)^{0.75}\right), \tag{20}$$

where $f_t$ denotes the feed intake on day $t$ calculated recursively using (19).

## B.3 Settlement pork price

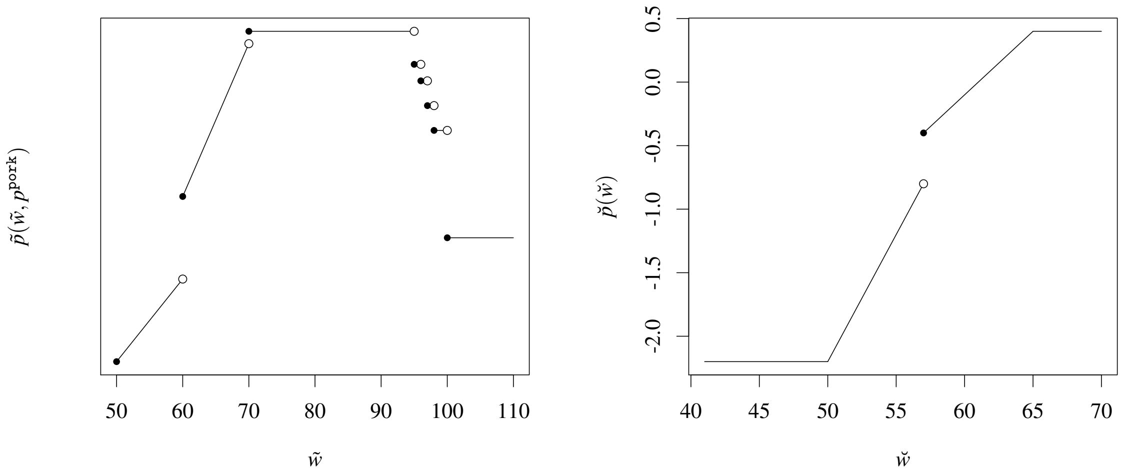
Consider the $k$th lightest pig at stage $n$ with carcass weight $\tilde{w}$ and leanness $\breve{w}$ at delivery. The settlement pork price, under Danish conditions, is the sum of two linear piecewise functions related to the price of the carcass and a bonus of the leanness:

$$p^{\mathtt{pork}}_{(k),n}(\tilde{w}, \breve{w}) = \tilde{p}(\tilde{w}, p^{\mathtt{pork}}) + \breve{p}(\breve{w}), \tag{21}$$

where $p^{\mathtt{pork}}$ is the current pork price at the market. Functions $\tilde{p}(\tilde{w}, p^{\mathtt{pork}})$ and $\breve{p}(\breve{w})$ correspond to the unit price of carcass and the bonus of leanness for 1 kilogram meat, respectively. A plot of each function is given in Figure 6.

Given the price structure, based on the Danish slaughter pig market[4], the unit price of 1 kilogram

[4] `http://www.danishcrown.dk/Ejer/Noteringer/Aktuel-svinenotering.aspx` (October 2015)




(a) Carcass weight.

(b) Leanness.

Figure 6: Price functions (DKK/kilogram) given carcass weight and leanness.

carcass is

$$\tilde{p}(\tilde{w}, p^{\text{pork}}) = \begin{cases} 0 & \tilde{w} < 50 \\ \frac{1}{9.9}(\tilde{w} - 50) + p^{\text{pork}} - 4 & 50 \leq \tilde{w} < 60 \\ \frac{1.85}{9.9}(\tilde{w} - 60) + p^{\text{pork}} - 2 & 60 \leq \tilde{w} < 70 \\ p^{\text{pork}} & 70 \leq \tilde{w} < 95 \\ p^{\text{pork}} - 0.2 & 95 \leq \tilde{w} < 96 \\ p^{\text{pork}} - 0.6 & 96 \leq \tilde{w} < 97 \\ p^{\text{pork}} - 0.9 & 97 \leq \tilde{w} < 98 \\ p^{\text{pork}} - 1.2 & 98 \leq \tilde{w} < 100 \\ p^{\text{pork}} - 2.5 & \tilde{w} \geq 100. \end{cases}$$

That is, the market pork price $p^{\text{pork}}$ may be interpreted as the maximum price of 1 kilogram carcass that can be obtained (when the carcass weight lies between 70 and 95 kilograms).

The bonus of leanness is calculated as

$$\breve{p}(\breve{w}) = \begin{cases} -2.2 & \breve{w} < 50 \\ 0.2(\breve{w} - 61) & 50 \leq \breve{w} < 57 \\ 0.1(\breve{w} - 61) & 57 \leq \breve{w} < 65 \\ 0.4 & \breve{w} \geq 65. \end{cases}$$

## B.4 Calculation of expected values

The calculations of the expected values (6)-(9) is rather complex due to the ordered random variables and the non-continuous functions $\tilde{p}(\tilde{w}, p^{\text{pork}})$ and $\breve{p}(\breve{w})$. However, the expectations can be calculated using simulation with a simple sorting procedure as described below.

**Step 0** For each pig $j = 1, \dots q^{\max}$, draw random vector $\alpha_j \sim N(0,\ V)$.

**Step 1** For each week $t$ and pig $j$, draw residual $\varepsilon_{j,t} \sim N(0,\ R)$ and find weight

$$w_{j,t} = X_t\beta + Z_t\alpha_j + \varepsilon_{j,t}.$$

Moreover, use the weights to find the daily growth $g$ during a week.

**Step 2** For each week $t$ and pig $j$, use (17) and (18) to find the carcass weight and leanness ($b$ days ahead), respectively. Moreover, use (19) to find the feed intake for the next $t = 7$ and $b$ days, i.e., (20) is calculated.

**Step 3** For each week $t$, pig $j$ and possible center point of pork price, calculate the settlement pork price (21).

**Step 4** For each week $t$, sort the obtained values of feed intake and settlement pork price in non-decreasing order of weight.

We run the simulation 10000 times to calculate average values of the feed intake and settlement pork price and next use the values to calculate the expected values (6)-(9).

# C Bayesian updating of SSMs

An SSM includes a set of observable and latent/unobservable continuous variables. The set of latent variables $\theta_{\{t=0,1,\dots\}}$ evolves over time using *system equation* (written using matrix notation)

$$\theta_t = G_t\theta_{t-1} + \omega_t, \tag{22}$$

where $\omega_t \sim N(0, W_t)$ is a random term and $G_t$ is a matrix of known values. We assume that the prior $\theta_0 \sim N(m_0, C_0)$ is given. Moreover, we have a set of observable variables $y_{\{t=1,2,\dots\}}$ (time-series data of prices) which are dependent on the latent variable using *observation equation*

$$y_t = F_t'\theta_t + \nu_t, \tag{23}$$

with $\nu_t \sim N(0, V_t)$. Here $F_t$ is the design matrix of system equations with known values and $F'$ denote the transpose of matrix $F$. The error sequences $\omega_t$ and $\nu_t$ are internally and mutually independent. Hence given $\theta_t$ we have that $y_t$ is independent of all other observations and in general the past and the future are independent given the present.

Let $\mathbb{D}_{t-1} = (y_1, \dots, y_{t-1}, m_0, C_0)$ denote the information available up to time $t-1$. Given the posterior of the latent variable at time $t-1$, we can use Bayesian updating (the Kalman filter) to update the distributions at time $t$ (West and Harrison, 1997, Thm 4.1).

**Theorem 1.** *Suppose that at time $t-1$ we have*

$$(\theta_{t-1} \mid \mathbb{D}_{t-1}) \sim N(m_{t-1}, C_{t-1}), \quad \textit{(posterior at time } t-1\textit{)}.$$

*then*

$$\begin{aligned}
(\theta_t \mid \mathbb{D}_{t-1}) &\sim N(a_t, R_t), \quad &&\textit{(one-step state distribution )}\\
(y_t \mid \mathbb{D}_{t-1}) &\sim N(f_t, Q_t), \quad &&\textit{(one-step forecast distribution)}\\
(\theta_t \mid \mathbb{D}_t) &\sim N(m_t, C_t), \quad &&\textit{(posterior at time } t\textit{)}
\end{aligned}$$

*where*

$$\begin{aligned}
a_t &= G_t m_{t-1}, & R_t &= G_t C_{t-1} G_t' + W_t \\
f_t &= F_t' a_t, & Q_t &= F_t' R_t F_t + V_t \\
e_t &= y_t - f_t, & A_t &= R_t F_t Q_t^{-1} \\
m_t &= a_t + A_t e_t, & C_t &= R_t - A_t Q_t A_t'.
\end{aligned}$$

Note that the means of the one-step state and forecast distributions, $a_t$ and $f_t$, only depend on $m_{t-1}$. Moreover variance $C_t$ only depends on the number of observations made, i.e., we can calculate it without knowing the observations $y_1, ..., y_t$. Similarly, we can find k-step conditional distributions.

**Theorem 2.** *Suppose that at time t we have*

$$(\theta_t \mid \mathbb{D}_t) \sim N(m_t, C_t), \quad \textit{(posterior at time t).}$$

*then*

$$\begin{aligned}
(y_{t+k}|m_t) &= (y_{t+k} \mid \mathbb{D}_t) \sim N(f_t(k), Q_t(k)), && \textit{(k-step forecast distribution)} \\
(m_{t+k}|m_t) &= (m_{t+k}|\mathbb{D}_t) \sim N\left(a_t(k), A_t(k) Q_t(k) A_t'(k)\right), && \textit{(k-step posterior mean distribution)}
\end{aligned}$$

*where* $f_t(k) = F_{t+k}' a_t(k)$, $Q_t(k) = F_{t+k}' R_t(k) F_{t+k} + V_{t+k}$ *and* $A_t(k) = R_t(k) F_{t+k} Q_t(k)^{-1}$ *which can be recursively calculated using*

$$\begin{aligned}
a_t(k) &= G_{t+k} a_t(k-1), \\
R_t(k) &= G_{t+k} R_t(k-1) G_{t+k}' + W_{t+k},
\end{aligned}$$

*with starting values* $a_t(0) = m_t$ *and* $R_t(0) = C_t$.

*Proof.* First, note that the probability distribution of $(y_{t+k}|\mathbb{D}_t)$ and the related proof have been given in West and Harrison (1997, Thm 4.2). Moreover, because $f_t(k)$ is a function of $m_t$, we have that $(y_{t+k} \mid \mathbb{D}_t) = (y_{t+k}|m_t)$.

Next, to find the probability distribution of $(m_{t+k}|\mathbb{D}_t)$, we use the similar procedure given in the proof of Theorem 4.2 in West and Harrison (1997, page 107-108). According to the repeated application of system equation in an SSM (West and Harrison, 1997, page 107), the $k$-step evolution of latent variable $\theta_t$ can be formulated as

$$\theta_{t+k} = G_{t+k}(k)\theta_t + \sum_{r=1}^{k} G_{t+k}(k-r)\omega_{t+r}, \tag{24}$$

where $G_{t+k}(r) = G_{t+k} G_{t+k-1} ... G_{t+k-r+1}$ for $r < k$, with $G_{t+k}(0) = I$. Now, using (22), (23) and (24), we can generate an SSM modelling the $k$-step evolution of $\theta_t$:

$$\text{Observation equation: } y_{t+k} = F_{t+k}' \theta_{t+k} + \nu_{t+k}$$

$$\text{System equation: } \theta_{t+k} = G_{t+k}(k)\theta_t + \sum_{r=1}^{k} G_{t+k}(k-r)\omega_{t+r}.$$

For this SSM we can use the general properties of Theorem 1 with $t-1$, $t$, $G_t$ and $\omega_t$ replaced with $t$, $t+k$, $G_{t+k}(k)$ and $\sum_{r=1}^{k} G_{t+k}(k-r)\omega_{t+r}$, respectively. Hence

$$m_{t+k} = a_t(k) + A_t(k) e_t(k),$$

where

$$a_t(k) = G_{t+k}(k)m_t, \quad e_t(k) = y_{t+k} - f_t(k).$$

Based on these equations and $(y_{t+k}|m_t) \sim N(f_t(k), Q_t(k))$, we have that

$$(m_{t+k}|m_t) \sim N(a_t(k), A_t(k)Q_t(k)A_t(k)'),$$

where based on the recursive equation for $G_{t+k}(r)$ and Theorem 1, we have that

$$\begin{aligned}
a_t(k) &= G_{t+k}(k)m_t = G_{t+k}a_t(k-1), \\
R_t(k) &= G_{t+k}(k)C_t G'_{t+k} + \sum_{r=1}^{k} G_{t+k}(k-r)W_{t+r}G_{t+k}(k-r)' \\
&= G_{t+k}R_t(k-1)G'_{t+k} + W_{t+k}, \\
A_t(k) &= R_t(k)F_{t+k}Q_t(k)^{-1}, \\
Q_t(k) &= F'_{t+k}R_t(k)F_{t+k} + V_{t+k}.
\end{aligned}$$

□