

MILP sensitivity analysis for the objective function coefficients*

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March 2021

1 Introduction

Sensitivity analysis in a mathematical programming problem determines how an optimal solution is affected by changes in the parameter values. For some problem classes, such as *linear programming (LP)*, the analysis can be carried out using post-optimal information (e.g., shadow prices or reduced costs). In fact, the region of parameter values for which the optimal solution remains the same can often be determined without any re-optimization. For other problem classes, including *mixed-integer linear programming (MILP)*, re-optimization is necessary. In this paper we consider sensitivity analysis in a MILP problem. More specifically, we investigate how an optimal solution is affected by changes in the objective function coefficient(s).

This paper presents a novel technique for sensitivity analysis of the objective function coefficients in a MILP based on multi-objective programming. In general, sensitivity analysis of the objective function coefficients in an LP comes down to solving a parametric linear programming problem. A connection between sensitivity analysis of the objective function coefficients in an LP and *multi-objective linear programming (MOLP)* has already been established. The paper by Yu and Zeleny [1976] considers a weighted sum problem that assigns non-negative weights to the objective functions. The authors observe that the nondominated points of the MOLP correspond to the optimal solutions to the weighted sum problem. Using this insight, they determine the weights for which a given non-dominated point corresponds to an optimal solution. Steuer [1986] likewise describes the connection between parametric analysis for the objective function coefficients of an LP and the nondominated points to a MOLP. The approach does not immediately generalize to integer LP (ILP). For an ILP, it is clear that if the convex hull of the feasible set is known, sensitivity analysis of the objective function coefficients reduces to that of an LP. In general, however, the set of nondominated points is not the same for a multi-objective integer linear program and the MOLP that arises by relaxing integrality.

In this paper, we consider the general MILP problem

$$\max\{\nu(x, y) \mid (x, y) \in \mathcal{X}\}$$

*Preprint of K.A. Andersen and T.K. Boomsma and L.R. Nielsen, *MILP Sensitivity Analysis for the Objective Function Coefficients* in *INFORMS Journal on Optimization* 2023, doi:10.1287/ijoo.2022.0078

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with objective function $\nu(x, y) = cx + hy$ with $c \in \mathbb{Q}^n$ and $h \in \mathbb{Q}^p$, where \mathbb{Q} is the set of rational numbers. The feasible set is $\mathcal{X} = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^p \mid Ax + Gy \leq b, x \geq 0, y \geq 0\}$ with $A \in \mathbb{Q}^{m \times n}$, $G \in \mathbb{Q}^{m \times p}$ and $b \in \mathbb{Q}^m$. Throughout the paper, we assume that $\mathcal{X} \neq \emptyset$ and that there exists an optimal solution (x^*, y^*) to Π with objective value $\nu^* = \nu(x^*, y^*) < +\infty$. For $K \subseteq \{1, 2, \dots, n\}$, our focus is the parameterized MILP obtained by replacing the objective function coefficients $c_k, k \in K$ by $c_k + \Delta_k, k \in K$

$$\max\{\nu_{(\Delta_k: k \in K)}(x, y) \mid (x, y) \in \mathcal{X}\},$$

with $\nu_{(\Delta_k: k \in K)}(x, y) = cx + hy + \sum_{k \in K} \Delta_k x_k$. We define the *sensitivity region* as the maximal region of simultaneous changes to the coefficients such that (x^*, y^*) remains optimal to Π , i.e.,

$$\Omega = \left\{ (\Delta_k : k \in K) \in \mathbb{R}^{|K|} \mid \nu_{(\Delta_k: k \in K)}(x, y) \leq \nu_{(\Delta_k: k \in K)}(x^*, y^*), (x, y) \in \mathcal{X} \right\}.$$

The region is maximal in the sense that for variations beyond this, the optimal solution to Π changes. The same analysis can be carried out for simultaneous variations in any coefficients of the vectors c and h .

The paper shows that the sensitivity region is determined by a subset of the vertices to the convex hull of the feasible set of the MILP. Although the convex hull is usually unknown, we show that these vertices correspond to nondominated points of a multi-objective optimization problem, and thus, the sensitivity region can be obtained from a subset of the nondominated points. Indeed if we consider sensitivity analysis on $|K|$ coefficients of the objective function, the sensitivity region is determined by solving a multi-objective MILP with $|K| + 1$ objectives which finds the extreme nondominated points adjacent to the nondominated point corresponding to the optimal solution of the MILP. For the special case of changes to a single objective function coefficient, we prove that the optimal solution of the MILP corresponds to a nondominated point of a bi-objective MILP and to obtain the sensitivity region it suffices to determine the two extreme adjacent nondominated points.

The multi-objective approach to sensitivity analysis applies to MILPs, including LPs and ILPs, and is as such, very general. Furthermore, it is independent of the solution method used for multi-objective programming. The paper gives numerical experiments providing preliminary computational experience of the proposed technique. Since there is a high interest in developing multi-objective solvers the computational performance will improve as multi-objective solvers become increasingly efficient (see e.g. Bökler, Parragh, Sinnl, and Tricoire [2021], Przybylski, Klamroth, and Lacour [2019]).

The paper is structured as follows. Section 2 gives a short literature review of approaches to sensitivity analysis in MILPs, and Section 3 provides a brief introduction to multi-objective analysis. In Section 4 we discuss sensitivity of MILP with respect to changes in a single objective function coefficient and in Subsection 4.3 we address the special case of MILP with binary variables. In Section 5 we extend our analysis to simultaneous changes in multiple objective function coefficients, and in Section 6 we give some preliminary experimental results. Section 7 summarizes our findings.

2 Literature review

To the best of our knowledge, sensitivity analysis for MILP was first addressed by Jensen [1968] who analyzed changes in the parameter values of a small instance. As pointed out by the author, sensitivity analysis for MILP is much more computationally demanding than for LP. Some general foundations for parametric analysis were suggested and discussed by Geoffrion and Nauss [1977] and Jenkins

[1990] who considered changes in the objective function coefficients and the right-hand side of the constraints. These papers also include extensive reviews of previous work.

One strand of the existing literature focuses on the branch and bound method. A post-optimality algorithm exploits the branching tree to obtain new primal feasible solutions or improve the dual bounds using the LP-relaxation. Such papers include those of Geoffrion and Nauss [1977], Ohtake and Nishida [1985], Piper and Zoltners [1975] and Roodman [1974]. This approach likewise covers changes in the objective function coefficients and the right-hand side. Other branch and bound approaches include Li and Ierapetritou [2007] and Oberdieck, Wittmann-Hohlbein, and Pistikopoulos [2014]. These papers contain illustrative examples but no experimental results.

A related line of postoptimality analysis for integer programming problems revolves around representations of optimal and near-optimal solutions. The authors Hadžić and Hooker [2006] propose a Multivalued Decision Diagram (MDD) to serve as a compact representation of the branching tree. The idea is further developed by Serra and Hooker [2017]. The authors demonstrate that an MDD can be used to answer a number of postoptimality questions, e.g., how to retrieve all feasible solutions with objective function values within a given distance to the optimal value.

Other papers apply cutting plane methods to estimate the convex hull of the feasible region. Examples are provided by Holm and Klein [1978] who investigate unit changes in the right-hand side and Klein and Holm [1979] who specify conditions for the current solution to remain optimal given changes in the objective function coefficients or the right-hand side.

With inspiration from LP, a substantial body of literature relies on duality theory. These papers study the *value function*, which parameterizes the optimal value of the problem by the right-hand side of the constraints [Guzelsoy and Ralphs, 2010]. Using duality, the value function of an LP problem can be expressed in terms of its shadow prices and shown to be piecewise linear and convex. Similarly, several papers have addressed MILP duality in terms of the value function, including Blair and Jeroslow [1977, 1979, 1982, 1984, 1985], Cook, Gerards, Schrijver, and Tardos [1986], Schrage and Wolsey [1985], Wolsey [1981] and Lasserre [2009]. On this basis, Tind and Wolsey [1981] and Wolsey [1981] apply subadditive price functions for sensitivity analysis. Moreover, by means of branch and bound, Schrage and Wolsey [1985] compute a piecewise linear value function that bounds the optimal value for a given perturbation of the right-hand side. A major challenge for MILP, however, is that the value function is in general non-convex and even discontinuous, see for instance Nemhauser and Wolsey [1988].

Another approach is based on *inference duality* and allows for changes in both the objective function coefficients, the right-hand side, and the coefficients of the constraint matrix [Dawande and Hooker, 2000]. This method, too, assumes that the MILP is solved by branch and bound. Given a permitted change to the objective function value, the authors set up a system of linear inequalities that account for perturbations of the parameters and have to be satisfied by every leaf node of the branching tree. Even so, their interval of change to a coefficient may only be a subset of its maximal interval. For more details, see Hooker [2009].

A few papers consider changes to a single objective function coefficient and obtain the interval for which a solution remains optimal by solving a sequence of MILP models, see Jenkins [1982] and Dua and Pistikopoulos [2000]. These methods are independent of the MILP solution method.

The recent paper by Charitopoulos, Papageorgiou, and Dua [2018] considers changes in both the objective function coefficients, the right hand side and the coefficients of the constraint matrix. It develops a novel solution method based on symbolic manipulation and semi-algebraic geometry. The method is illustrated with a number of examples and also contains experimental results for rather small instances. This paper also contains a summary of developments in multi-parametric programming theory.

We believe that our paper is the first to obtain *maximal* sensitivity regions for changes to *multiple* objective function coefficients in general MILPs.

The multi-objective approach to sensitivity analysis is independent of the solution method used for multi-objective optimization. That is, the computational performance will improve as multi-objective solvers become increasingly efficient. In the recent years there have been a high interest in developing multi-objective solvers for finding the nondominated set. For instance, for multi-objective ILP see Forget, Gadegaard, and Nielsen [2022], Kirlik and Sayin [2014], Tamby and Vanderpooten [2021] and for multi-objective MILP see Doğan, Lokman, and Köksalan [2021], Eichfelder and Warnow [2021], Pal and Charkhgard [2019], Stidsen and Andersen [2018], Stidsen, Andersen, and Dammann [2014]. Other paper consider algorithms for finding a subset [Bökler et al., 2021, Przybylski et al., 2019].

3 Multi-objective optimization

Consider the *multi-objective mixed-integer linear programming* (MO-MILP) problem given by

$$\max \{z(x, y) \in \mathbb{R}^q \mid (x, y) \in \mathcal{X}\},$$

which maps a feasible *solution* $(x, y) \in \mathcal{X}$ into an *objective point* $z(x, y) = (z_1(x, y), \dots, z_q(x, y))$ with $z_i(x, y) = c^i x + h^i y$, $i \in \{1, \dots, q\}$, $c^i \in \mathbb{Q}^n$, $h^i \in \mathbb{Q}^p$. We refer to \mathcal{X} as the feasible set in *solution space* and $\mathcal{Z} = \{z(x, y) \in \mathbb{R}^q \mid (x, y) \in \mathcal{X}\}$ as the corresponding feasible set in *objective space*. We say that two feasible solutions $(x^1, y^1), (x^2, y^2)$ are *equivalent* if they map into the same objective point, i.e. $z(x^1, y^1) = z(x^2, y^2)$. Feasible solutions are generally compared in terms of the following binary relation in objective space. For objective points $z^1, z^2 \in \mathbb{R}^q$,

$$z^1 \succ z^2 \quad \text{if and only if} \quad z_i^1 \geq z_i^2, \quad i = 1, \dots, q \text{ and } z^1 \neq z^2.$$

A point z^2 is said to be *dominated* by z^1 if $z^1 \succ z^2$. On the basis of this, an 'optimal' solution to Π_{MO} is defined by the concept of Pareto optimality or efficiency.

Definition 1. A vector $(x, y) \in \mathcal{X}$ is called Pareto optimal or *efficient* if $\nexists(\hat{x}, \hat{y}) \in \mathcal{X} : z(\hat{x}, \hat{y}) \succ z(x, y)$. The corresponding objective vector $z(x, y)$ is said to be a *nondominated* point.

The *efficient set* \mathcal{X}_E is a set of feasible efficient solutions such that all $(x, y) \in \mathcal{X} \setminus \mathcal{X}_E$ are either dominated by or equivalent to a point in \mathcal{X}_E . Moreover, the *set of nondominated points* $\mathcal{Z}_N = \{z(x, y) \in \mathcal{Z} \mid (x, y) \in \mathcal{X}_E\}$ is the corresponding image set in objective space¹. For $\mathcal{Z}_N \neq \emptyset$, we further define

$$\mathcal{Z}^{\leq} = \text{conv}(\mathcal{Z}_N + \{z \in \mathbb{R}^q : z \leq 0\}),$$

where $+$ denotes the Minkowski sum and $\text{conv}(\cdot)$ denotes the convex hull. The set \mathcal{Z}^{\leq} is a polyhedron, and thus, can be characterized by its vertices and rays or by its facets. Given a vertex of \mathcal{Z}^{\leq} , we refer to the adjacent vertices as those lying on the same facet of \mathcal{Z}^{\leq} . We use this to introduce the notions of supported and extreme nondominated points.

Definition 2. A point $z \in \mathcal{Z}_N$ is a *supported* nondominated point if it is on the boundary of \mathcal{Z}^{\leq} ; otherwise it is *unsupported*. A supported nondominated point z is *extreme* if it is a vertex of \mathcal{Z}^{\leq} ; otherwise it is *nonextreme*.

¹There may be several efficient sets \mathcal{X}_E that differ only by the inclusion of equivalent solutions. All efficient sets, however, map into the unique set of nondominated points \mathcal{Z}_N . In this paper, we assume that the *efficient set* \mathcal{X}_E is complete but of minimal cardinality (for each nondominated point we only include one efficient solution).

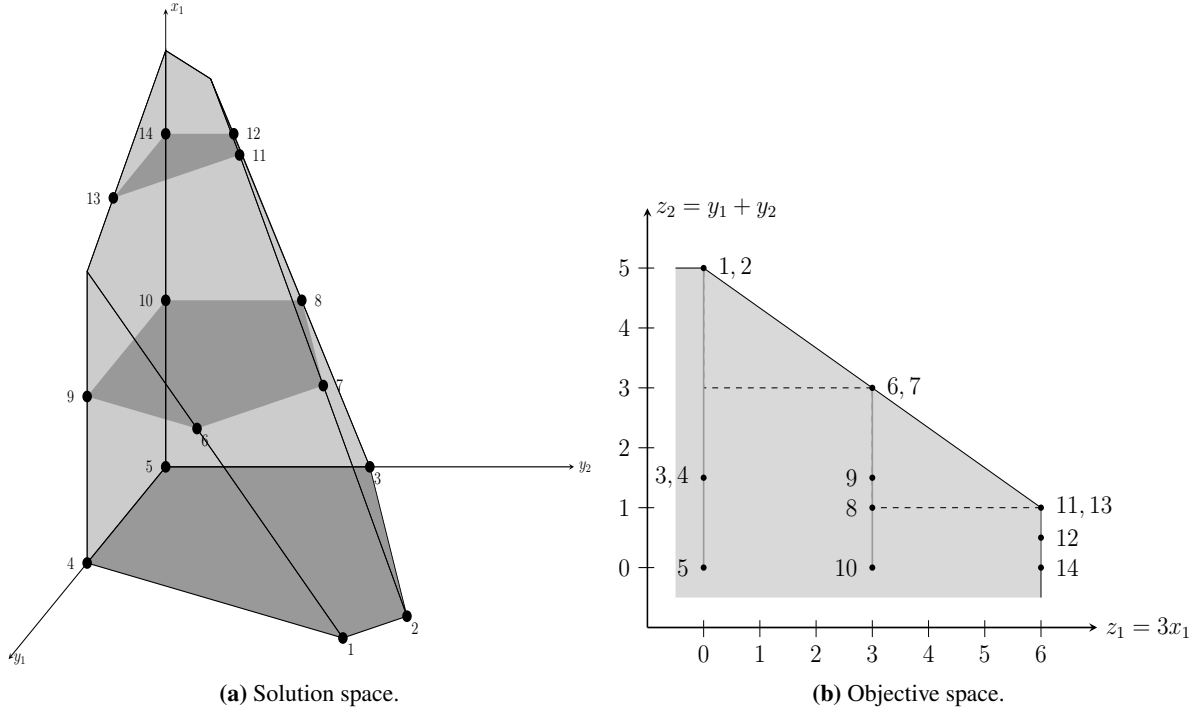


Figure 1: Solution space (Figure 1a) and objective space (Figure 1b) of the bi-objective MILP in Example 1. The dark gray regions define the feasible set in solution space. A vertex in solution space corresponds to a point in objective space with the same number.

For bi-objective MILP problems ($q = 2$), the extreme nondominated points include the so-called *upper left* (UL) and *lower right* (LR) points in \mathcal{Z}_N . We define

$$z_2^{UL} = \max\{z_2 \mid (z_1, z_2) \in \mathcal{Z}_N\} \quad \text{and} \quad z_1^{LR} = \max\{z_1 \mid (z_1, z_2) \in \mathcal{Z}_N\},$$

such that the upper left point z^{UL} and the lower right point z^{LR} are as follows

$$z^{UL} = \{(z_1, z_2) \in \mathcal{Z}_N \mid z_2 = z_2^{UL}\} \quad \text{and} \quad z^{LR} = \{(z_1, z_2) \in \mathcal{Z}_N \mid z_1 = z_1^{LR}\}.$$

Example 1. Consider the bi-objective MILP

$$\max\{(3x_1, y_1 + y_2) \mid (x, y) \in \mathcal{X}\}$$

with \mathcal{X} defined by the constraints

$$\begin{aligned} 2x_1 + y_1 + y_2 &\leq 5 \\ 2y_1 - y_2 &\leq 3 \\ x_1 - y_1 + 2y_2 &\leq 3 \\ y_1, y_2 &\geq 0, x_1 \in \mathbb{Z}_+ \end{aligned} \tag{1}$$

Figure 1a illustrates the solution space. The feasible set of the LP-relaxation is depicted in light gray, whereas the feasible set \mathcal{X} of the MILP consists of the union of three polytopes that are depicted in dark gray. The vertices of the polytopes are numbered from 1 to 14.

Figure 1b depicts the objective space. The feasible set \mathcal{Z} in objective space is the union of the line segments $\text{LS}(z^1, z^5)$, $\text{LS}(z^6, z^{10})$ and $\text{LS}(z^{11}, z^{14})$, where $\text{LS}(z^i, z^j)$ denotes the line segment between the objective points z^i and z^j . The nondominated set is

$$\mathcal{Z}_N = \{z^1, z^6, z^{11}\},$$

e.g. corresponding to the (minimal cardinality) set of efficient solutions

$$\mathcal{X}_E = \{(x^1, y^1), (x^6, y^6), (x^{11}, y^{11})\},$$

and the light gray region illustrates \mathcal{Z}^\leq . A vertex of \mathcal{Z}^\leq corresponds to a vertex in the solution space with the same number. Note that some vertices in the solution space correspond to the same vertex in \mathcal{Z}^\leq . For instance, vertices number 11 and 13 correspond to the solutions $(x^{11}, y^{11}) = (2, 1/3, 2/3)$ and $(x^{13}, y^{13}) = (2, 1, 0)$, respectively, and the objective point $z^{11} = z^{13} = (6, 1)$. The objective points z^1 and z^{11} are extreme nondominated points whereas z^6 is a supported nonextreme nondominated point. The upper left point is $z^{UL} = z^1$ and the lower right point is $z^{LR} = z^{11}$. \square

4 Varying a single objective function coefficient

This section determines the sensitivity region for changes to a single objective function coefficient, i.e., $|K| = 1$. We show that it suffices to inspect a subset of the feasible solutions to the MILP.

We distinguish between two cases; the case of a current objective function coefficient different from zero and the case of a coefficient equal to zero, see Subsection 4.1 and Subsection 4.2, respectively. In the first case, we first assume that the sign of the coefficient remains the same upon a change. The feasible points of interest are efficient solutions to a bi-objective MILP with one objective function being the contribution of the variable/coefficient under consideration and the other objective being the remaining contribution to the objective of the MILP. The idea is that the efficient solutions define the tradeoff between the two objectives. We next assume that the sign of the coefficient changes. Now, the subset of the feasible points further includes the efficient set of another bi-objective MILP with the first objective function having opposite sign.

In Subsection 4.3 we consider special cases, in particular, a change to an objective function coefficient for which the corresponding variable is binary.

4.1 The case $c_k \neq 0$

Define the parametrized MILP obtained by replacing the coefficient c_k by $c_k + \Delta$ for varying Δ

$$\max\{\nu_\Delta(x, y) \mid (x, y) \in \mathcal{X}\},$$

with $\nu_\Delta(x, y) = cx + hy + \Delta x_k$. It is easy to see that the sensitivity region is a convex set and therefore an interval. We let lb be its lower bound and ub be its upper bound such that

$$[lb, ub] = \{\Delta \in \mathbb{R} \mid \nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*), (x, y) \in \mathcal{X}\},$$

that is, $\Delta \in [lb, ub]$ if and only if

$$\Delta \geq \frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \quad (2)$$

for $x_k < x_k^*$, $\forall (x, y) \in \mathcal{X}$, and

$$\Delta \leq \frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \quad (3)$$

for $x_k > x_k^*$, $\forall (x, y) \in \mathcal{X}$. For $x_k = x_k^*$ there are no restrictions on Δ .

Define the interval $[lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$ of a subset $\hat{\mathcal{X}}$ of \mathcal{X} as

$$lb(\hat{\mathcal{X}}) = \sup \left\{ \frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \mid (x, y) \in \hat{\mathcal{X}}, x_k < x_k^* \right\}, \text{ and}$$

$$ub(\hat{\mathcal{X}}) = \inf \left\{ \frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \mid (x, y) \in \hat{\mathcal{X}}, x_k > x_k^* \right\},$$

assuming that supremum and infimum of the empty set equal $-\infty$ and ∞ , respectively. Clearly, $lb(\hat{\mathcal{X}}) \leq 0$ and $ub(\hat{\mathcal{X}}) \geq 0$. Combining the definition with equations (2)-(3), we obtain the following lemma.

Lemma 1. *The sensitivity region is $[lb, ub] = [lb(\mathcal{X}), ub(\mathcal{X})]$. Moreover, if $\hat{\mathcal{X}} \subseteq \mathcal{X}$ then*

$$lb(\hat{\mathcal{X}}) \leq lb(\mathcal{X}) \leq 0 \leq ub(\mathcal{X}) \leq ub(\hat{\mathcal{X}}).$$

Lemma 1 states the obvious fact that $\Delta = 0$ is within the sensitivity region as Π_Δ reduces to Π . Furthermore, for $\hat{\mathcal{X}} \subseteq \mathcal{X}$, the interval $[lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$ contains the sensitivity region. Our goal is to identify subsets of \mathcal{X} for which this interval is precisely the sensitivity region. We illustrate the idea in Example 2.

Example 2. Consider the MILP

$$\max\{3x_1 + y_1 + y_2 \mid (x, y) \in \mathcal{X}\},$$

where the feasible set \mathcal{X} is defined in (1) and illustrated in Figure 1a. An optimal solution is $(x^*, y^*) = (x^{11}, y^{11}) = (2, \frac{1}{3}, \frac{2}{3})$ (vertex number 11) with objective $\nu^* = \nu(x^*, y^*) = 7$. We aim to determine the sensitivity region for changes to each of the objective function coefficients $c_1 = 3$, $h_1 = 1$ and $h_2 = 1$.

The sensitivity regions can be determined from the set of vertices of the convex hull of \mathcal{X} . Hence, if set

$$\hat{\mathcal{X}} = \{(x^1, y^1), \dots, (x^{14}, y^{14})\},$$

(see Figure 1a), then $[lb, ub] = [lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$. The sensitivity regions found using the vertices are shown in Table 1. For example, the sensitivity region of changes to coefficient h_1 is $[-\frac{3}{2}, 0]$, where the lower bound is determined by vertex 12 and the upper bound is determined by vertex 13.

Now, consider the bi-objective MO-MILPs

$$\max\{(3x_1, y_1 + y_2) \mid (x, y) \in \mathcal{X}\} \quad (\text{coefficient } c_1), \text{ and} \quad (4)$$

$$\max\{(y_1, 3x_1 + y_2) \mid (x, y) \in \mathcal{X}\} \quad (\text{coefficient } h_1), \text{ and} \quad (5)$$

$$\max\{(y_2, 3x_1 + y_1) \mid (x, y) \in \mathcal{X}\} \quad (\text{coefficient } h_2), \quad (6)$$

obtained by letting the first objective be the contribution of the variable/coefficient under consideration (as indicated in the parenthesis) and the second objective be the remaining contribution to the objective of the MILP.

Table 1: Sensitivity regions for the coefficients c_1 , h_1 and h_2 . Computations are based on either the set of vertices ($\hat{\mathcal{X}}$) or the efficient set (\mathcal{X}_E).

Coeff. ^a	Fig. ^b	Using $\hat{\mathcal{X}}$				Using \mathcal{X}_E				
		$lb(\hat{\mathcal{X}})$	$ub(\hat{\mathcal{X}})$	arg lb^c	arg ub^d	\mathcal{Z}_N^e	$lb(\mathcal{X}_E)$	$ub(\mathcal{X}_E)$	arg lb^f	arg ub^g
$c_1 = 3$	1b	-1	∞	1, 2, 6, 7	none	1, 6, 11	-1	∞	1, 6	none
$h_1 = 1$	2a	$-\frac{3}{2}$	0	12	13	LS(11,13), LS(6,7), LS(1,2)	$-\infty$	0	none	13
$h_2 = 1$	2b	0	1	13	2, 7	LS(11,13), LS(6,7), LS(1,2)	0	1	13	2, 7

^a: Objective function coefficient under consideration and its current value. ^b: The figure showing the objective space for the MO-MILP solved. ^c: Indices of the vertices in Figure 1a that finds $lb(\hat{\mathcal{X}})$. ^d: Indices of the vertices in Figure 1a that finds $ub(\hat{\mathcal{X}})$. ^e: Nondominated point indices (see the figure with the objective space). LS(i, j) is the line segment between the two points with indices i and j . ^f: Indices of the nondominated points in the objective space that finds $lb(\mathcal{X}_E)$. ^g: Indices of the nondominated points in the objective space that finds $ub(\mathcal{X}_E)$.

The nondominated points of (4) are illustrated in Figure 1b and the nondominated points of (5) and (6) are illustrated in Figure 2. Table 1 provides the corresponding nondominated sets \mathcal{Z}_N , the bounds $lb(\mathcal{X}_E)$ and $ub(\mathcal{X}_E)$ and the nondominated points determining the bounds. Note that the vertices in $\hat{\mathcal{X}}$ determining a sensitivity region are efficient solutions, except for MO-MILP (5), where vertex number 12 is not an efficient solution. In this case, the nondominated point corresponding to the optimal solution is the upper left point. \square

Example 2 indicates that it may suffice to consider the efficient solutions (or nondominated points) of a bi-objective problem to determine the sensitivity region. Below, we show that this generally holds when the sign of the objective function coefficient remains the same.

Define the bi-objective problem Π_{MO}^+ with objective vector

$$z^+(x, y) = (z_1(x, y), z_2(x, y)) = (c_k x_k, \sum_{\substack{i=1 \\ i \neq k}}^n c_i x_i + h y). \quad (7)$$

and feasible set \mathcal{X} . Notice that $\nu(x, y) = z_1(x, y) + z_2(x, y)$. Denote by \mathcal{X}_E^+ the efficient set of Π_{MO}^+ , defining the tradeoff between the two objectives. Now, we can select $(x^*, y^*) \in \mathcal{X}_E^+$, and hence, $\mathcal{X}_E^+ \neq \emptyset$. To see this, assume that $(x^*, y^*) \notin \mathcal{X}_E^+$ for all optimal solutions. Then there exists $(\bar{x}, \bar{y}) \in \mathcal{X}$ such that $c_k \bar{x}_k \geq c_k x_k^*$ and $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i x_i^* + h y^*$, and at least one of the inequalities is strict. This implies that $c \bar{x} + h \bar{y} > c x^* + h y^*$, a contradiction to the optimality of (x^*, y^*) to the problem Π .

Now, the interval $[lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$ contains the sensitivity region. Lemma 2 shows that if $c_k > 0$ (resp. $c_k < 0$) and $c_k + \Delta \geq 0$ (resp. $c_k + \Delta \leq 0$) for all $\Delta \in [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$, then \mathcal{X}_E^+ defines the sensitivity region. Otherwise, we may only determine a subset of the sensitivity region.

Lemma 2. Consider the bi-objective problem Π_{MO}^+ with efficient set \mathcal{X}_E^+ .

If $c_k > 0$, define

$$lb^+ = \max(lb(\mathcal{X}_E^+), -c_k), \quad \text{and} \quad ub^+ = ub(\mathcal{X}_E^+).$$

Then $lb \leq lb^+ \leq ub^+ = ub$. Moreover, if $lb^+ = lb(\mathcal{X}_E^+)$ then $lb^+ = lb$.

If $c_k < 0$, define

$$lb^+ = lb(\mathcal{X}_E^+), \quad \text{and} \quad ub^+ = \min(ub(\mathcal{X}_E^+), -c_k).$$

Then $lb = lb^+ \leq ub^+ \leq ub$. Moreover, if $ub^+ = ub(\mathcal{X}_E^+)$ then $ub^+ = ub$.

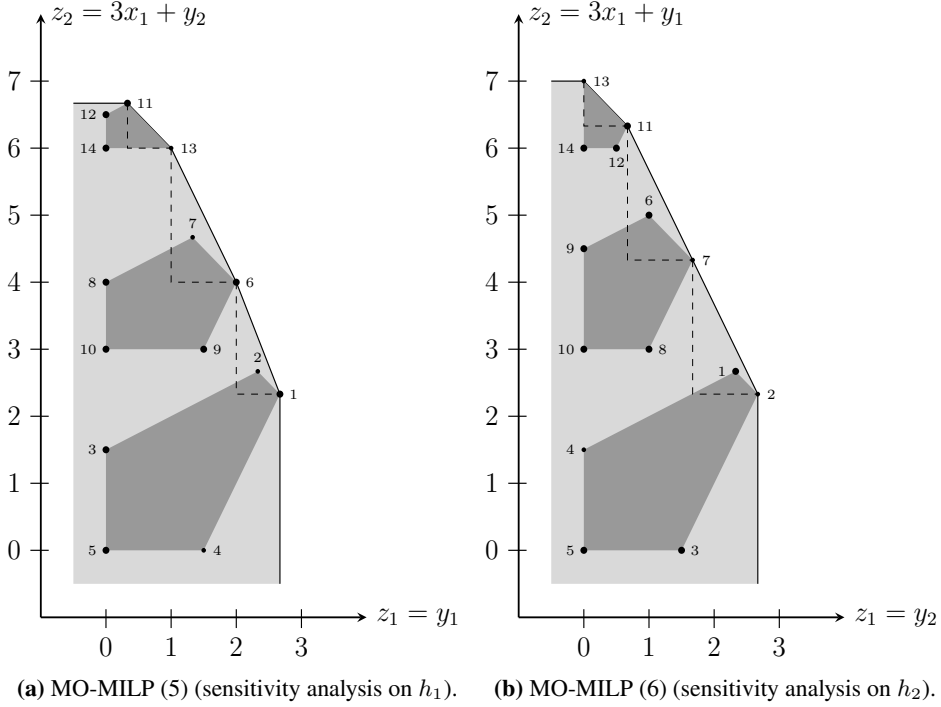


Figure 2: Objective space of the MO-MILPs (5) and (6). The objective space of MO-MILP (4) (sensitivity analysis on c_1) is illustrated in Figure 1b. The dark gray regions define the feasible sets in objective space. A vertex in objective space corresponds to the vertex in solution space with the same number, see Figure 1a.

Proof. Using Lemma 1 it is sufficient to show that $[lb^+, ub^+] \subseteq [lb(\mathcal{X}), ub(\mathcal{X})]$. Conversely, now assume that $\exists \Delta : \Delta \in [lb^+, ub^+] \setminus [lb(\mathcal{X}), ub(\mathcal{X})]$ implying that $\exists (\hat{x}, \hat{y}) \in \mathcal{X} : \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$. Note that since $\Delta \in [lb^+, ub^+] \subseteq [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$, we have $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*)$, $(x, y) \in \mathcal{X}_E^+$. Hence, $(\hat{x}, \hat{y}) \notin \mathcal{X}_E^+$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}_E^+$ such that (i) $c_k \bar{x}_k \geq c_k \hat{x}_k$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i \hat{x}_i + h \hat{y}$. We multiply (i) by $(c_k + \Delta)/c_k \geq 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) \geq \nu_\Delta(\hat{x}, \hat{y})$. But then $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, contradicting $\Delta \in [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$. \square

If the sign of the objective function coefficient changes, we define another bi-objective problem Π_{MO}^- with objective vector

$$z^-(x, y) = (-z_1(x, y), z_2(x, y)) = (-c_k x_k, \sum_{\substack{i=1 \\ i \neq k}}^n c_i x_i + h y).$$

and feasible set \mathcal{X} . Notice that the sign of the first objective function is now the opposite. Denote by \mathcal{X}_E^- the efficient set of Π_{MO}^- . With the opposite sign the first objective may be unbounded, that is $\sup\{-c_k x_k \mid (x, y) \in \mathcal{X}\} = +\infty$, and then $\mathcal{X}_E^- = \emptyset$. However, if \mathcal{X} is compact this situation cannot occur. Lemma 3 shows that if $c_k > 0$ (resp. $c_k < 0$) and the lower bound lb^+ (resp. ub^+) of Lemma 2 is determined by $-c_k$, then the set \mathcal{X}_E^- may be required to determine the remainder of the sensitivity region.

Lemma 3. Consider the bi-objective problem Π_{MO}^- with efficient set \mathcal{X}_E^- .

If $c_k > 0$ and $lb(\mathcal{X}_E^+) < -c_k$, define

$$lb^- = \begin{cases} lb(\mathcal{X}_E^-), & \text{if } \mathcal{X}_E^- \neq \emptyset, \\ -c_k, & \text{otherwise} \end{cases} \quad \text{and } ub^- = -c_k.$$

Then $lb = lb^- \leq ub^- \leq ub$.

If $c_k < 0$ and $ub(\mathcal{X}_E^+) > -c_k$, define

$$lb^- = -c_k, \text{ and } ub^- = \begin{cases} ub(\mathcal{X}_E^-), & \text{if } \mathcal{X}_E^- \neq \emptyset, \\ -c_k, & \text{otherwise} \end{cases}$$

Then $lb \leq lb^- \leq ub^- = ub$.

Proof. The proof follows the same lines as the proof of Lemma 2, see Appendix A. \square

The combination of Lemmas 2 and 3 is summarized in the following theorem.

Theorem 1. Consider the problems Π_{MO}^+ and Π_{MO}^- with efficient sets \mathcal{X}_E^+ and \mathcal{X}_E^- , respectively. For $c_k \neq 0$, the sensitivity region $[lb, ub]$ is given by

$$lb = \begin{cases} -c_k, & \text{if } c_k > 0, lb(\mathcal{X}_E^+) < -c_k \text{ and } \mathcal{X}_E^- = \emptyset, \\ lb(\mathcal{X}_E^-), & \text{if } c_k > 0, lb(\mathcal{X}_E^+) < -c_k \text{ and } \mathcal{X}_E^- \neq \emptyset, \\ lb(\mathcal{X}_E^+), & \text{otherwise,} \end{cases}$$

and

$$ub = \begin{cases} -c_k, & \text{if } c_k < 0, ub(\mathcal{X}_E^+) > -c_k \text{ and } \mathcal{X}_E^- = \emptyset, \\ ub(\mathcal{X}_E^-), & \text{if } c_k < 0, ub(\mathcal{X}_E^+) > -c_k \text{ and } \mathcal{X}_E^- \neq \emptyset, \\ ub(\mathcal{X}_E^+), & \text{otherwise.} \end{cases}$$

Theorem 1 holds for all efficient sets \mathcal{X}_E^+ and \mathcal{X}_E^- of the bi-objective problems. As an alternative to using the efficient solutions (in solution space) in dimension \mathbb{R}^{n+p} , it may be convenient to use the nondominated points (in objective space) in dimension \mathbb{R}^2 to determine the sensitivity region. Let

$$\mathcal{Z}_N^+ = \{z^+(x, y) \mid (x, y) \in \mathcal{X}_E^+\},$$

denote the nondominated set of Π_{MO}^+ and let

$$\mathcal{Z}^- = \{z^+(x, y) \mid (x, y) \in \mathcal{X}_E^-\},$$

denote the efficient set of Π_{MO}^- projected into the objective space of $z^+(x, y)$. Notice that $z^+(x, y) \in \mathcal{Z}^-$ if and only if $z^-(x, y)$ is a nondominated point of Π_{MO}^- . Given $(z_1^*, z_2^*) = z^+(x^*, y^*)$ and a set $\hat{\mathcal{Z}}$ of objective points, define

$$lb(\hat{\mathcal{Z}}) = \sup \left\{ \frac{c_k(z_1^* + z_2^* - z_1 - z_2)}{z_1 - z_1^*} \mid (z_1, z_2) \in \hat{\mathcal{Z}}, \frac{z_1}{c_k} < \frac{z_1^*}{c_k} \right\}, \text{ and}$$

$$ub(\hat{\mathcal{Z}}) = \inf \left\{ \frac{c_k(z_1^* + z_2^* - z_1 - z_2)}{z_1 - z_1^*} \mid (z_1, z_2) \in \hat{\mathcal{Z}}, \frac{z_1}{c_k} > \frac{z_1^*}{c_k} \right\},$$

assuming that the supremum and the infimum of the empty set equal $-\infty$ and ∞ , respectively.

Corollary 1. For $c_k \neq 0$,

$$lb(\mathcal{X}_E^+) = lb(\mathcal{Z}_N^+), ub(\mathcal{X}_E^+) = ub(\mathcal{Z}_N^+), lb(\mathcal{X}_E^-) = lb(\mathcal{Z}^-), ub(\mathcal{X}_E^-) = ub(\mathcal{Z}^-),$$

i.e. the sensitivity region can be found by replacing \mathcal{X}_E^+ with \mathcal{Z}_N^+ and \mathcal{X}_E^- with \mathcal{Z}^- in Theorem 1.

Proof. Insert $z_1^* + z_2^* = \nu(x^*, y^*)$, $z_1 + z_2 = \nu(x, y)$, $z_1/c_k = x_k$ and $z_1^*/c_k = x_k^*$ into the definitions of $lb(\hat{\mathcal{X}})$ and $ub(\hat{\mathcal{X}})$. \square

Example 3. We revisit Example 2. By Theorem 1 and Corollary 1, a sensitivity region can be determined from efficient solutions or nondominated points. The sensitivity region of the coefficients c_1 and h_2 can be found using the nondominated set \mathcal{Z}_N^+ of Π_{MO}^+ with objectives (4) and (6), respectively. The sensitivity region of the coefficient h_1 is computed using both \mathcal{Z}_N^+ and \mathcal{Z}^- (i.e. the line segment between objective points z^{11} and z^{12} in Figure 2a), as the lower bound is $lb(\mathcal{Z}_N^+) = -\infty$. Observe that the sensitivity region of the coefficient c_1 can be determined using the extreme nondominated point $z^1 = (0, 5)$ in Figure 1b (see also Table 1). Similarly, the sensitivity region of the objective function coefficient h_2 can be determined using extreme points $z^7 = (\frac{3}{2}, \frac{9}{2})$ and $z^{13} = (1, 6)$, as observed in Figure 2b. Finally, the sensitivity region of the objective function coefficient h_1 is determined using points $z^{12} = (\frac{1}{2}, 6)$ (with $(-\frac{1}{2}, 6)$ in \mathcal{Z}^-) and $z^{13} = (0, 7)$ (in \mathcal{Z}_N^+) illustrated in Figure 2a, where $(-\frac{1}{2}, 6)$ and $(0, 7)$ are extreme nondominated points. \square

Example 3 suggests that to determine the sensitivity region, it suffices to consider the extreme nondominated points. Accordingly, let $\mathcal{Z}_{N,e}^+$ denote the set of extreme points of

$$\mathcal{Z}_{\leq}^+ = \text{conv}(\mathcal{Z}_N^+ + \{z \in \mathbb{R}^q : z \leq 0\}),$$

and \mathcal{Z}_e^- denote the set of extreme points of

$$\mathcal{Z}_{\leq}^- = \text{conv}(\mathcal{Z}^- + \{z \in \mathbb{R}^q : z \leq 0\}).$$

Corollary 2. The sensitivity region can be found by replacing \mathcal{Z}_N^+ with $\mathcal{Z}_{N,e}^+$ and \mathcal{Z}^- with \mathcal{Z}_e^- in Corollary 1.

Proof. Note that $\mathcal{Z}_{N,e}^+ \subseteq \mathcal{Z}_N^+$. Assume that there exists a Δ such that $\exists(\hat{z}_1, \hat{z}_2) \in \mathcal{Z}_N^+ \setminus \mathcal{Z}_{N,e}^+ : \hat{z}_1(c_k + \Delta)/c_k + \hat{z}_2 = \hat{z}_\Delta > z_\Delta^* = z_1^*(c_k + \Delta)/c_k + z_2^*$ and $z_1(c_k + \Delta)/c_k + z_2 = z_\Delta \leq z_\Delta^* = z_1^*(c_k + \Delta)/c_k + z_2^*$, $(z_1, z_2) \in \mathcal{Z}_{N,e}^+$. Since $(\hat{z}_1, \hat{z}_2) \in \mathcal{Z}_{\leq}^+ \setminus \mathcal{Z}_{N,e}^+$ there are some $(\bar{z}_1^1, \bar{z}_2^1), (\bar{z}_1^2, \bar{z}_2^2) \in \mathcal{Z}_{N,e}^+$ and $\lambda \in (0, 1)$ such that $\hat{z}_1 = \lambda \bar{z}_1^1 + (1 - \lambda) \bar{z}_1^2$ and $\hat{z}_2 = \lambda \bar{z}_2^1 + (1 - \lambda) \bar{z}_2^2$. Since, however, $(\bar{z}_1^1, \bar{z}_2^1), (\bar{z}_1^2, \bar{z}_2^2) \in \mathcal{Z}_{N,e}^+$, we have that (i) $\bar{z}_1^1(c_k + \Delta)/c_k + \bar{z}_2^1 \leq z_1^*(c_k + \Delta)/c_k + z_2^*$ and (ii) $\bar{z}_1^2(c_k + \Delta)/c_k + \bar{z}_2^2 \leq z_1^*(c_k + \Delta)/c_k + z_2^*$. We multiply (i) by λ and (ii) by $1 - \lambda$ and add the them to obtain $\hat{z}_1(c_k + \Delta)/c_k + \hat{z}_2 \leq z_1^*(c_k + \Delta)/c_k + z_2^*$, which is a contradiction. The other case is treated similarly. \square

Example 4. Again, revisit Example 2. Table 2 lists the extreme nondominated points. Notice that the sensitivity regions are determined by at most two of the extreme points. For example, the sensitivity region for h_1 is determined by the two points z^{12} and z^{13} (see Figure 2a) and the sensitivity region for h_2 is determined by z^7 and z^{13} (see Figure 2b). For c_1 , the permissible increase is $+\infty$, and the lower bound can be found using z^1 (see Figure 1b). \square

Table 2: Sensitivity analysis for the coefficients c_1, h_1 and h_2 . The projections of the efficient sets into objective space, $\mathcal{Z}_{N,e}^+$ and \mathcal{Z}_e^- .

Coeff.	Sensitivity analysis on c_1				Sensitivity analysis on h_1				Sensitivity analysis on h_2			
	$z_1 = 3x_1$	$z_2 = y_1 + y_2$	$\mathcal{Z}_{N,e}^+$	\mathcal{Z}_e^-	$z_1 = y_1$	$z_2 = 3x_1 + y_2$	$\mathcal{Z}_{N,e}^+$	\mathcal{Z}_e^-	$z_1 = y_2$	$z_2 = 3x_1 + y_1$	$\mathcal{Z}_{N,e}^+$	\mathcal{Z}_e^-
1	0	5	✓	✓	$\frac{8}{3}$	$\frac{7}{3}$	✓		$\frac{7}{3}$	$\frac{8}{3}$		
2	0	5	✓	✓	$\frac{7}{3}$	$\frac{8}{3}$			$\frac{8}{3}$	$\frac{7}{3}$	✓	
3	0	$\frac{3}{2}$			0	$\frac{3}{2}$			$\frac{3}{2}$	0		
4	0	$\frac{3}{2}$			$\frac{3}{2}$	0			0	$\frac{3}{2}$		
5	0	0			0	0			0	0		
6	3	3	✓		2	4	✓		1	5		
7	3	3	✓		$\frac{4}{3}$	$\frac{14}{3}$			$\frac{5}{3}$	$\frac{13}{3}$	✓	
8	3	1			0	4			1	3		
9	3	$\frac{5}{3}$			$\frac{3}{2}$	3			0	$\frac{9}{2}$		
10	3	0			0	3			0	3		
11	6	1	✓		$\frac{1}{3}$	$\frac{20}{3}$	✓	✓	$\frac{2}{3}$	$\frac{19}{3}$	✓	
12	6	$\frac{1}{2}$			0	$\frac{13}{2}$		✓	$\frac{1}{2}$	6		
13	6	1	✓		1	6	✓		0	7	✓	✓
14	6	0			0	6			0	6		

Example 4 further demonstrates that the extreme points used to determine the sensitivity region lie on the same facets of $\text{conv}(\mathcal{Z}_{N,e}^+ \cup \mathcal{Z}_e^-)$ as the objective point corresponding to the optimal solution, i.e. they are the vertices adjacent to $z^+(x^*, y^*)$. For instance, in Figure 2a the vertices z^{12} and z^{13} are adjacent to $z^+(x^*, y^*) = z^{11}$ and in Figure 2b the vertices z^7 and z^{13} are adjacent to z^{11} . However, $z^+(x^*, y^*)$ may not be an extreme point (but a convex combination of two extreme points).

On this basis, Theorem 2 provides a precise characterization of the sensitivity region. To illustrate, assume that $c_k > 0$ ($c_k < 0$ is similar) and order the points in the set $\mathcal{Z}_e^- \cup \mathcal{Z}_{N,e}^+ \cup \{z^+(x^*, y^*)\}$ according to increasing first coordinate. If $z^+(x^*, y^*)$ is neither the first nor the last point in this ordering, then the maximal sensitivity region can be found using the two adjacent points to $z^+(x^*, y^*)$. If $z^+(x^*, y^*)$ corresponds to the last point then the objective function coefficient can increase to $+\infty$, because there is no feasible point setting an upper limit, and the lower limit of a decrease can be found using the next to last point. If $z^+(x^*, y^*)$ corresponds to the first point, two possibilities exist. If $\mathcal{Z}_e^- \neq \emptyset$, then $\mathcal{Z}_e^- = \{z^+(x^*, y^*)\}$, and the objective function coefficient can decrease to $-\infty$, because there is no feasible point setting a lower limit. If, on the other hand, $\mathcal{Z}_e^- = \emptyset$, then the lower limit is equal to $-c_k$.

Theorem 2. Assume that $c_k \neq 0$. Consider the ordered set

$$\mathcal{Z}_e^- \cup \mathcal{Z}_{N,e}^+ \cup \{z^+(x^*, y^*)\} = \{z^1, z^2, \dots, z^J, \dots, z^I\}$$

with $z_1^i < z_1^{i+1}$ for $i = 1, \dots, I-1$ and $\mathcal{Z}_{N,e}^+ \cup \{z^+(x^*, y^*)\} = \{z^J, \dots, z^I\}$. Define $J \leq i^* \leq I$, such that $z^{i^*} = z^+(x^*, y^*)$. The sensitivity region $[lb, ub]$ is given by

$$lb = \begin{cases} -\infty, & \text{if } i^* = I, c_k < 0, \\ lb(\{z^{i^*+1}\}), & \text{if } i^* < I, c_k < 0, \\ -\infty, & \text{if } i^* = 1, \mathcal{Z}_e^- \neq \emptyset, c_k > 0, \\ -c_k, & \text{if } i^* = 1, \mathcal{Z}_e^- = \emptyset, c_k > 0, \\ lb(\{z^{i^*-1}\}), & \text{if } i^* > 1, c_k > 0, \end{cases} \quad ub = \begin{cases} \infty, & \text{if } i^* = 1, \mathcal{Z}_e^- \neq \emptyset, c_k < 0, \\ -c_k, & \text{if } i^* = 1, \mathcal{Z}_e^- = \emptyset, c_k < 0, \\ ub(\{z^{i^*-1}\}), & \text{if } i^* > 1, c_k < 0, \\ \infty, & \text{if } i^* = I, c_k > 0, \\ ub(\{z^{i^*+1}\}), & \text{if } i^* < I, c_k > 0. \end{cases}$$

Proof. It is well-known from multi-objective optimization, that the piecewise linear function obtained by connecting the points z^1, \dots, z^I is concave, see also Figure 2a and Table 2. If $\mathcal{Z}_e^- \neq \emptyset$ and $z^J \in \mathcal{Z}_e^-$, the function is strictly increasing between the points z^1, \dots, z^J and strictly decreasing between the points z^J, \dots, z^I . If $\mathcal{Z}_e^- \neq \emptyset$ and $z^J \notin \mathcal{Z}_e^-$, the function is strictly increasing between the points z^1, \dots, z^{J-1} , it is constant between z^{J-1} and z^J , and strictly decreasing between the points z^J, \dots, z^I . In particular, the slope between the points z^i and z^{i+1} is strictly larger than the slope between the points z^{i+1} and z^{i+2} , for $i = 1, 2, \dots, I - 2$. We prove the theorem for $c_k > 0$. The proof for $c_k < 0$ is similar.

Notice that

$$\frac{z_2^{i^*} - z_2^j}{z_1^{i^*} - z_1^j} = -\frac{1}{c_k} \cdot \frac{c_k(z_1^{i^*} + z_2^{i^*} - z_1^j - z_2^j)}{z_1^j - z_1^{i^*}} - 1, \quad j \in \{1, \dots, I\} \setminus \{i^*\}.$$

Thus, the strictly decreasing slopes imply that

$$\frac{c_k(z_1^{i^*} + z_2^{i^*} - z_1^j - z_2^j)}{z_1^j - z_1^{i^*}} < \frac{c_k(z_1^{i^*} + z_2^{i^*} - z_1^{j+1} - z_2^{j+1})}{z_1^{j+1} - z_1^{i^*}}, \quad j, j+1 \in \{1, \dots, I\} \setminus \{i^*\}.$$

By comparing this expression with the lower and upper bounds $lb(\mathcal{Z}_{N,e}^+)$ and $ub(\mathcal{Z}_{N,e}^+)$, it should be clear that the supremum is attained at the point z^{i^*-1} if it exists, and that the infimum is attained at the point z^{i^*+1} if it exists. Therefore, if $i^* < I$ then $ub(\mathcal{Z}_{N,e}^+) = ub(\{z^{i^*+1}\})$ and if $i^* > J$ then $lb(\mathcal{Z}_{N,e}^+) = lb(\{z^{i^*-1}\}) > -c_k$. Moreover, if $i^* = I$ then $z_1^{i^*} > z_1^j$ for all $j \in \{J, \dots, I\} \setminus \{i^*\}$, and so, the infimum is not attained and $ub(\mathcal{Z}_{N,e}^+) = +\infty$. If $i^* = J$, then $z_1^{i^*} < z_1^j$ for all $j \in \{J, \dots, I\} \setminus \{i^*\}$, the supremum is not attained and $lb(\mathcal{Z}_{N,e}^+) = -\infty < -c_k$.

By Theorem 1, if $i^* < I$ then $ub = ub(\mathcal{Z}_{N,e}^+) = ub(\{z^{i^*+1}\})$ and if $i^* = I$ then $ub = ub(\mathcal{Z}_{N,e}^+) = +\infty$. Also, if $i^* > J$ then $lb = lb(\mathcal{Z}_{N,e}^+) = lb(\{z^{i^*-1}\})$. Consider $i^* = J$. If $\mathcal{Z}_e^- = \emptyset$, then $J = 1$ and $lb = -c_k$. Consider $\mathcal{Z}_e^- \neq \emptyset$. If $J > 1$ then the supremum of the lower bound $lb(\mathcal{Z}_e^-)$ is attained at the point z^{i^*-1} , and so, $lb = lb(\mathcal{Z}_e^-) = lb(\{z^{i^*-1}\})$. If $J = 1$ then the supremum is not attained and $lb = lb(\mathcal{Z}_e^-) = -\infty$. \square

4.2 The case $c_k = 0$

Define the bi-objective programs, $\Pi_{\text{MO}}^{+,0}$ with objective vector

$$z^+(x, y) = (z_1(x, y), z_2(x, y)) = (x_k, \sum_{\substack{i=1 \\ i \neq k}}^n c_i x_i + hy)$$

and $\Pi_{\text{MO}}^{-,0}$ with objective vector

$$z^-(x, y) = (-z_1(x, y), z_2(x, y)) = (-x_k, \sum_{\substack{i=1 \\ i \neq k}}^n c_i x_i + hy),$$

both with feasible set \mathcal{X} . Notice, that efficient set $\mathcal{X}_E^{+,0}$ of $\Pi_{\text{MO}}^{+,0}$ does not depend on the objective function coefficient of x_k in the first objective, as long as it is positive. Similarly, the efficient set $\mathcal{X}_E^{-,0}$ of $\Pi_{\text{MO}}^{-,0}$ is independent of the objective function coefficient of x_k in the first objective, as long as it is negative. Note also that we may have $\mathcal{X}_E^{+,0} = \emptyset$ and $\mathcal{X}_E^{-,0} = \emptyset$.

We obtain the following extension to Theorem 1.

Proposition 1. Consider the problems $\Pi_{MO}^{+,0}$ and $\Pi_{MO}^{-,0}$ with efficient sets $\mathcal{X}_E^{+,0}$ and $\mathcal{X}_E^{-,0}$, respectively. For $c_k = 0$, the sensitivity region $[lb, ub]$ is given by

$$lb = \begin{cases} lb(\mathcal{X}_E^{-,0}), & \text{if } \mathcal{X}_E^{-,0} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad ub = \begin{cases} ub(\mathcal{X}_E^{+,0}), & \text{if } \mathcal{X}_E^{+,0} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See Appendix A. □

4.3 Special cases

Recall that if $c_k \neq 0$ then $z^+(x^*, y^*)$ is a nondominated point to the bi-objective problem Π_{MO}^+ . If this is the only nondominated point of Π_{MO}^+ and Π_{MO}^- , the value of c_k has no effect on the optimal solution.

Corollary 3. Assume that $c_k \neq 0$. If $|\mathcal{Z}_N^+ \cup \mathcal{Z}^-| = 1$, then $lb = -\infty$ and $ub = +\infty$.

Proof. If $|\mathcal{Z}_N^+ \cup \mathcal{Z}^-| = 1$ then $\mathcal{Z}_{N,e}^+ = \mathcal{Z}_e^- = \{z^+(x^*, y^*)\}$, and hence, $i = J = I$ in Theorem 2. □

If x_k is binary, then $1 \leq |\mathcal{Z}_N^+| \leq 2$, that is, we have at most two nondominated points of Π_{MO}^+ . One nondominated point is $z^+(x^*, y^*)$ and the other candidate is $\hat{z}^+ = (\hat{z}_1^+, \hat{z}_2^+)$ with $\hat{z}_1^+ = c_k(1 - x_k^*)$ and

$$\hat{z}_2^+ = \sup \left\{ \sum_{\substack{i=1 \\ i \neq k}}^n c_i x_i + h y \mid (x, y) \in \mathcal{X}, x_k = 1 - x_k^* \right\}.$$

Now, we obtain the following.

Corollary 4. Assume that $c_k \neq 0$ and x_k is binary. If $\hat{z}_2^+ = -\infty$, then $lb = -\infty$ and $ub = \infty$. Otherwise, $z^+(x^*, y^*) \in \mathcal{Z}_{N,e}^+$ and

1. if $x_k^* = 1$ then $lb = \hat{z}_1^+ + \hat{z}_2^+ - \nu(x^*, y^*)$ and $ub = \infty$.
2. if $x_k^* = 0$ then $ub = \nu(x^*, y^*) - \hat{z}_1^+ - \hat{z}_2^+$ and $lb = -\infty$.

Proof. If $\hat{z}_2^+ = -\infty$, then $|\mathcal{Z}_N^+ \cup \mathcal{Z}^-| = 1$ and the first result follows from Corollary 3. Otherwise, $|\mathcal{Z}_N^+ \cup \mathcal{Z}^-| = |\mathcal{Z}_{N,e}^+ \cup \mathcal{Z}_e^-| = 2$ and $z^+(x^*, y^*) \in \mathcal{Z}_{N,e}^+$ since it is nondominated under objective (7). If $x_k^* = 1$, then $i^* = I$ in Theorem 2 and the lower and upper bound becomes $lb_\Delta = \hat{z}_1^+ + \hat{z}_2^+ - \nu(x^*, y^*)$ and $ub_\Delta = \infty$, respectively. If $x_k^* = 0$, then $i^* = 1$ in Theorem 2 and the lower and upper bound becomes $ub_\Delta = \nu(x^*, y^*) - \hat{z}_1^+ - \hat{z}_2^+$ and $lb_\Delta = -\infty$, respectively. □

Corollary 4 is similar to Proposition 2.3 in Geoffrion and Nauss [1977] and to Theorems 1 and 3 in Pisinger and Saidi [2017].

Example 5. To illustrate the results of this section, we use an example presented by Schrage and Wolsey [1985].

$$\begin{aligned} \max \nu(x) &= 77x_1 + 6x_2 + 3x_3 + 6x_4 + 33x_5 + 13x_6 + 110x_7 + 21x_8 + 47x_9 \\ \text{s.t.} \quad &774x_1 + 76x_2 + 22x_3 + 42x_4 + 21x_5 + 760x_6 + 818x_7 + 62x_8 + 785x_9 \leq 1500 \\ &67x_1 + 27x_2 + 794x_3 + 53x_4 + 234x_5 + 32x_6 + 792x_7 + 97x_8 + 435x_9 \leq 1500 \\ &x_j \in \{0, 1\}, j = 1, 2, \dots, 9. \end{aligned}$$

The optimal solution is $x^* = (x_1^*, \dots, x_9^*) = (0, 1, 0, 1, 1, 0, 1, 1, 0)$. The optimal value is $\nu(x^*) = 176$.

k	1	2	3	4	5	6	7	8	9
c_k	77	6	3	6	33	13	110	21	47
lb	$-\infty$	-6	$-\infty$	-6	-33	$-\infty$	-30	-21	$-\infty$
ub	30	$+\infty$	30	$+\infty$	$+\infty$	94	$+\infty$	$+\infty$	63

Table 3: Lower bounds (lb) and upper bounds (ub) of the sensitivity region for c_k .

To determine the sensitivity interval for c_1 , note that $z^+(x^*, y^*) = (0, 176)$, and that $\hat{z}^+ = (77, 69)$. By Corollary 4, $ub = 176 - 77 - 69 = 30$ and $lb = -\infty$. The sensitivity interval for Δ is therefore $(-\infty, 30]$. For changes to c_2 , note that $z^+(x^*, y^*) = (6, 170)$, and that $\hat{z}^+ = (0, 170)$. By Corollary 4, $lb = 0 + 170 - 176 = -6$ and $ub = +\infty$, and the sensitivity interval for Δ is $[-6, \infty)$. The other sensitivity intervals are given in Table 3. \square \square

5 Simultaneous variations of multiple objective function coefficients

We extend our analysis to determine the sensitivity region for simultaneous changes to two or more objective function coefficients. For a single objective function coefficient, it suffices to inspect at most two sets of efficient solutions, each to a bi-objective problem. For simultaneous changes to two objective function coefficients, we have to examine at most four efficient sets, corresponding to the four combinations of the signs of the two coefficients, and each corresponding to a tri-objective problem. The results generalize to simultaneous variations of multiple objective function coefficients. In particular, for changes in $|K|$ objective function coefficients we have to consider at most $2^{|K|}$ sets of efficient solutions, likewise corresponding to the $2^{|K|}$ combinations of their signs, and each corresponding to a multi-objective problem with $|K| + 1$ objective functions.

In Subsection 5.2 we show that the 100% rule [Bradley, Hax, and Magnanti, 1977] in linear programming also holds for MILPs.

5.1 Varying two objective function coefficients

The parameterized MILP Π_{Δ_k, Δ_l} obtained by varying the coefficients c_k and c_l is:

$$\max\{\nu_{(\Delta_k, \Delta_l)}(x, y) \mid (x, y) \in \mathcal{X}\},$$

where $\nu_{(\Delta_k, \Delta_l)}(x, y) = cx + hy + \Delta_k x_k + \Delta_l x_l$. The sensitivity region is then

$$\Omega = \{(\Delta_k, \Delta_l) \in \mathbb{R}^2 \mid \nu_{(\Delta_k, \Delta_l)}(x, y) \leq \nu_{(\Delta_k, \Delta_l)}(x^*, y^*), (x, y) \in \mathcal{X}\}.$$

For a subset $\hat{\mathcal{X}}$ of \mathcal{X} , we define the region

$$\Omega(\hat{\mathcal{X}}) = \{(\Delta_k, \Delta_l) \in \mathbb{R}^2 \mid \nu_{(\Delta_k, \Delta_l)}(x, y) \leq \nu_{(\Delta_k, \Delta_l)}(x^*, y^*), (x, y) \in \hat{\mathcal{X}}\}.$$

If $\hat{\mathcal{X}} \neq \emptyset$, then $\Omega(\hat{\mathcal{X}})$ is a convex set. If $\hat{\mathcal{X}} = \emptyset$, we define $\Omega(\hat{\mathcal{X}}) = \emptyset$.

The extension of Lemma 1 follows immediately.

Lemma 4. *The sensitivity region is $\Omega = \Omega(\mathcal{X})$. Moreover, if $\hat{\mathcal{X}} \subseteq \mathcal{X} \setminus \emptyset$, then $\Omega(\hat{\mathcal{X}}) \supseteq \Omega(\mathcal{X})$.*

Assume that $c_k, c_l \neq 0$ and define the tri-objective problems $\Pi_{\text{MO}}^{++}, \Pi_{\text{MO}}^{+-}, \Pi_{\text{MO}}^{-+}$ and Π_{MO}^{--} with the four distinct objective vectors

$$\begin{aligned} z^{++}(x, y) &= (z_1(x, y), z_2(x, y), z_3(x, y)) = (c_k x_k, c_l x_l, \sum_{\substack{i=1 \\ i \neq k, l}}^n c_i x_i + h y), \text{ and} \\ z^{+-}(x, y) &= (z_1(x, y), -z_2(x, y), z_3(x, y)) = (c_k x_k, -c_l x_l, \sum_{\substack{i=1 \\ i \neq k, l}}^n c_i x_i + h y), \text{ and} \\ z^{-+}(x, y) &= (-z_1(x, y), z_2(x, y), z_3(x, y)) = (-c_k x_k, c_l x_l, \sum_{\substack{i=1 \\ i \neq k, l}}^n c_i x_i + h y), \text{ and} \\ z^{--}(x, y) &= (-z_1(x, y), -z_2(x, y), z_3(x, y)) = (-c_k x_k, -c_l x_l, \sum_{\substack{i=1 \\ i \neq k, l}}^n c_i x_i + h y), \end{aligned}$$

and feasible set \mathcal{X} . Denote by \mathcal{X}_E^{++} the efficient set to Π_{MO}^{++} , and similarly for $\mathcal{X}_E^{+-}, \mathcal{X}_E^{-+}$ and \mathcal{X}_E^{--} .

To determine the sensitivity region $\Omega(\mathcal{X})$, we further define the sets

$$\begin{aligned} \Omega^{++} &= \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{++}) \mid \text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(c_k), 0\}, \text{sgn}(c_l + \Delta_l) \in \{\text{sgn}(c_l), 0\}\}, \text{ and} \\ \Omega^{+-} &= \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{+-}) \mid \text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(c_k), 0\}, \text{sgn}(c_l + \Delta_l) \in \{-\text{sgn}(c_l), 0\}\}, \text{ and} \\ \Omega^{-+} &= \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{-+}) \mid \text{sgn}(c_k + \Delta_k) \in \{-\text{sgn}(c_k), 0\}, \text{sgn}(c_l + \Delta_l) \in \{\text{sgn}(c_l), 0\}\}, \text{ and} \\ \Omega^{--} &= \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{--}) \mid \text{sgn}(c_k + \Delta_k) \in \{-\text{sgn}(c_k), 0\}, \text{sgn}(c_l + \Delta_l) \in \{-\text{sgn}(c_l), 0\}\}, \end{aligned}$$

where $\text{sgn}(\cdot)$ is the signum function. If $c_k > 0$ then $\text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(c_k), 0\}$ means that $c_k + \Delta_k \geq 0$, and if $c_k < 0$ it means $c_k + \Delta_k \leq 0$.

We can now formulate our theorem.

Theorem 3. *The sensitivity region is $\Omega = \Omega^{++} \cup \Omega^{+-} \cup \Omega^{-+} \cup \Omega^{--}$.*

Proof. By Lemma 4 it is sufficient to show that $\Omega^{++} \cup \Omega^{+-} \cup \Omega^{-+} \cup \Omega^{--} \subseteq \Omega(\mathcal{X})$.

Assume on the contrary that $\exists(\Delta_k, \Delta_l) : (\Delta_k, \Delta_l) \in (\Omega^{++} \cup \Omega^{+-} \cup \Omega^{-+} \cup \Omega^{--}) \setminus \Omega(\mathcal{X})$. Since $(\Delta_k, \Delta_l) \notin \Omega(\mathcal{X})$, $\exists(\hat{x}, \hat{y}) \in \mathcal{X} : \nu_{(\Delta_k, \Delta_l)}(\hat{x}, \hat{y}) > \nu_{(\Delta_k, \Delta_l)}(x^*, y^*)$. There are four cases: 1) $(\Delta_k, \Delta_l) \in \Omega^{++}$, 2) $(\Delta_k, \Delta_l) \in \Omega^{+-}$, 3) $(\Delta_k, \Delta_l) \in \Omega^{-+}$ and 4) $(\Delta_k, \Delta_l) \in \Omega^{--}$. We consider the first two. Cases 3) and 4) can be proven similarly.

1) Since $(\Delta_k, \Delta_l) \in \Omega^{++} \subseteq \Omega(\mathcal{X}_E^{++})$, we have that $(\hat{x}, \hat{y}) \notin \mathcal{X}_E^{++}$. This implies that there exists some $(\bar{x}, \bar{y}) \in \mathcal{X}_E^{++}$ such that (i) $c_k \bar{x}_k \geq c_k \hat{x}_k$ and (ii) $c_l \bar{x}_l \geq c_l \hat{x}_l$ and (iii) $\sum_{i \neq k, l} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k, l} c_i \hat{x}_i + h \hat{y}$. We multiply (i) by $(c_k + \Delta_k)/c_k \geq 0$ and (ii) by $(c_l + \Delta_l)/c_l \geq 0$ and add them to (iii) to obtain $\nu_{(\Delta_k, \Delta_l)}(\bar{x}, \bar{y}) > \nu_{(\Delta_k, \Delta_l)}(\hat{x}, \hat{y}) > \nu_{(\Delta_k, \Delta_l)}(x^*, y^*)$. This contradicts that $z_{\Delta_k, \Delta_l}(x, y) \leq z_{\Delta_k, \Delta_l}(x^*, y^*)$, $(x, y) \in \mathcal{X}_E^{++}$.

2) By the same reasoning as above, there exists some $(\bar{x}, \bar{y}) \in \mathcal{X}_E^{+-}$ such that (i) $c_k \bar{x}_k \geq c_k \hat{x}_k$ and (ii) $-c_l \bar{x}_l \geq -c_l \hat{x}_l$ and (iii) $\sum_{i \neq k, l} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k, l} c_i \hat{x}_i + h \hat{y}$. We multiply (i) by $(c_k + \Delta_k)/c_k \geq 0$ and (ii) by $(c_l + \Delta_l)/c_l \leq 0$ and add them to (iii) to obtain $\nu_{(\Delta_k, \Delta_l)}(\bar{x}, \bar{y}) > \nu_{(\Delta_k, \Delta_l)}(\hat{x}, \hat{y}) > \nu_{(\Delta_k, \Delta_l)}(x^*, y^*)$, which is again a contradiction. \square

Since Ω is a convex set, we have that if $\Delta_k \neq -c_k$ and $\Delta_l \neq -c_l$ for all $(\Delta_k, \Delta_l) \in \Omega^{++}$, then $\Omega = \Omega^{++}$ and it is sufficient to consider the tri-objective problem Π_{MO}^{++} .

As in Corollary 1 and Corollary 2, the set Ω^{++} can be expressed in terms of the extreme nondominated points in objective space. Let $(z_1^*, z_2^*, z_3^*) = z^{++}(x^*, y^*)$ and denote by $\mathcal{Z}_{N,e}^{++}$ the set of extreme nondominated points to Π_{MO}^{++} . Then,

$$\Omega^{++} = \{(\Delta_k, \Delta_l) \in \mathbb{R}^2 \mid \frac{\Delta_k}{c_k}(z_1 - z_1^*) + \frac{\Delta_l}{c_l}(z_2 - z_2^*) \leq z_1^* + z_2^* + z_3^* - z_1 - z_2 - z_3, \\ (z_1, z_2, z_3) \in \mathcal{Z}_{N,e}^{++}, \text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(c_k), 0\}, \text{sgn}(c_l + \Delta_l) \in \{\text{sgn}(c_l), 0\}\}.$$

The sets Ω^{+-} , Ω^{-+} and Ω^{--} can likewise be expressed in terms of extreme nondominated points.

For the general analysis of variations of three or more objective function coefficients, see Appendix B.

We illustrate the results on two examples.

Example 6. We return to the example presented by Schrage and Wolsey [1985].

To determine the region of simultaneous changes to c_k and c_l , note that there are at most four nondominated points, that is, $z^{++}(x^*, y^*) = (c_k x_k^*, c_l x_l^*, \sum_{i \neq k,l} c_i x_i^*)$, $\hat{z}^{+-} = (c_k x_k^*, c_l(1 - x_l^*), \hat{z}_3^{+-})$, $\hat{z}^{-+} = (c_k(1 - x_k^*), c_l x_l^*, \hat{z}_3^{-+})$ and $\hat{z}^{--} = (c_k(1 - x_k^*), c_l(1 - x_l^*), \hat{z}_3^{--})$ with

$$\hat{z}_3^{+-} = \sup \left\{ \sum_{\substack{i=1 \\ i \neq k,l}}^9 c_i x_i^* \mid (x, y) \in \mathcal{X}, x_k = x_k^*, x_l = 1 - x_l^* \right\}, \text{ and} \\ \hat{z}_3^{-+} = \sup \left\{ \sum_{\substack{i=1 \\ i \neq k,l}}^9 c_i x_i^* \mid (x, y) \in \mathcal{X}, x_k = 1 - x_k^*, x_l = x_l^* \right\}, \text{ and} \\ \hat{z}_3^{--} = \sup \left\{ \sum_{\substack{i=1 \\ i \neq k,l}}^9 c_i x_i^* \mid (x, y) \in \mathcal{X}, x_k = 1 - x_k^*, x_l = 1 - x_l^* \right\}.$$

For changes to c_1 and c_3 , the nondominated points are $\{(0, 0, 176), (0, 3, 86), (77, 0, 99), (77, 3, 66)\}$. The sensitivity region is $\Omega = \{(\Delta_1, \Delta_3) \in \mathbb{R}^2 \mid \Delta_1 \leq 33, \Delta_3 \leq 87, \Delta_1 + \Delta_3 \leq 30\}$. For changes to c_7 and c_9 , the nondominated points are $\{(0, 0, 146), (0, 47, 66), (77, 0, 99)\}$ (for $x_7 = x_9 = 1$, the problem is infeasible) and the sensitivity region is $\Omega = \{(\Delta_7, \Delta_9) \in \mathbb{R}^2 \mid \Delta_7 \geq -30, -\Delta_7 + \Delta_9 \leq 63\}$. \square

Example 7. Return to the MILP in Example 2.

We consider simultaneous changes of the objective function coefficients c_1 and h_1 . We start by finding the set Ω^{++} . The efficient solutions are $\mathcal{X}_E^{++} = \{(x^i, y^i), i = 1, 6, 11\}$, and hence, $\Omega^{++} = \{(\Delta_1, \Delta_2) \in \mathbb{R}^2 \mid -\Delta_1 + \Delta_2 \leq 1, \Delta_1 \geq -3, \Delta_2 \leq 0, \Delta_2 \geq -1\}$. Notice that if $(\Delta_1, \Delta_2) \in \Omega^{++}$, we cannot have that $\Delta_1 = -3$. Thus, $\Omega^{-+} = \Omega^{--} = \emptyset$. However, we may have $\Delta_2 = -1$. Therefore, we proceed to find the set Ω^{+-} . The efficient solutions are $\mathcal{X}_E^{+-} = \{(x^i, y^i), i = 2, 7, 11, 12\}$, and so, $\Omega^{+-} = \{(\Delta_1, \Delta_2) \in \mathbb{R}^2 \mid -\Delta_1 + \Delta_2 \leq 1, \Delta_2 \leq -1, \Delta_2 \geq -1.5\}$. It follows that $\Omega = \Omega^{++} \cup \Omega^{+-} = \{(\Delta_1, \Delta_2) \in \mathbb{R}^2 \mid -\Delta_1 + \Delta_2 \leq 1, \Delta_2 \leq 0, \Delta_2 \geq -1.5\}$. \square

5.2 The 100% rule

According to the 100% rule in linear programming, the solution remains optimal for simultaneous changes in the objective function coefficients provided that the sum of relative changes is at most 100%, see Bradley et al. [1977]. We extend this result to MILP.

Table 4: Inequalities with weights.

Inequalities		Weight
$cx + hy$	$\leq cx^* + hy^*$	$1 - \sum_{i \in K_{lb}^c \cup K_{lb}^h} \frac{\Delta_i}{lb_i} - \sum_{i \in K_{ub}^c \cup K_{ub}^h} \frac{\Delta_i}{ub_i}$
$cx + hy + lb_i x_i$	$\leq cx^* + hy^* + lb_i x_i^*$, for $i \in K_{lb}^c$	$\frac{\Delta_i}{lb_i}$
$cx + hy + lb_i y_i$	$\leq cx^* + hy^* + lb_i y_i^*$, for $i \in K_{lb}^h$	$\frac{\Delta_i}{lb_i}$
$cx + hy + ub_i x_i$	$\leq cx^* + hy^* + ub_i x_i^*$, for $i \in K_{ub}^c$	$\frac{\Delta_i}{ub_i}$
$cx + hy + ub_i y_i$	$\leq cx^* + hy^* + ub_i y_i^*$, for $i \in K_{ub}^h$	$\frac{\Delta_i}{ub_i}$

For $k \in \{1, 2, \dots, n, n+1, \dots, n+p\}$, denote by Δ_k the change to coefficient k of the vector $c \in \mathbb{R}^n$ or the vector $h \in \mathbb{R}^p$ and let lb_k and ub_k be the lower and upper bounds of its sensitivity interval. Denote by $K_{lb}^c, K_{ub}^c \subseteq \{1, 2, \dots, n\}$ the two disjoint sets of indices for which the coefficients of c decrease respectively increase, and similarly, denote by $K_{lb}^h, K_{ub}^h \subseteq \{n+1, n+2, \dots, n+p\}$ the sets for which the coefficients of h decrease respectively increase. The 100% rule can be stated as follows.

Theorem 4. Assume that

$$\sum_{i \in K_{lb}^c \cup K_{lb}^h} \frac{\Delta_i}{lb_i} + \sum_{i \in K_{ub}^c \cup K_{ub}^h} \frac{\Delta_i}{ub_i} \leq 1.$$

Then $(\Delta_1, \Delta_2, \dots, \Delta_n, \Delta_{n+1}, \dots, \Delta_{n+p})$ is in the sensitivity region.

Proof. Consider $(x, y) \in \mathcal{X}$. We weigh the inequalities in Table 4 and add them to obtain

$$\sum_{i=1}^n (c_i + \Delta_i)x_i + \sum_{i=n+1}^{n+p} (h_i + \Delta_i)y_i \leq \sum_{i=1}^n (c_i + \Delta_i)x_i^* + \sum_{i=n+1}^{n+p} (h_i + \Delta_i)y_i^*.$$

□

6 Numerical experiments

We carry out sensitivity analysis on two problems from the literature; a pure ILP for capital budgeting and a MILP formulation of a production lot sizing problem. Our numerical experiments serve to provide preliminary computational experience of the proposed technique. Since it is independent of the solution method used the computational performance will improve as multi-objective solvers become increasingly efficient.

For bi-objective mixed-integer linear programming, we determine the extreme adjacent nondominated points using a modification of the first phase of the so-called two-phase method, see Przybylski, Gandibleux, and Ehrgott [2008]. For general MO-MILP, we use the code developed by Bökler and Mutzel [2015].

The two-phase method and modifications to this have been implemented in C++ using Visual Studio C++ 2015 and ILOG CPLEX Studio 12.7. The tests are performed on an x-64 based Fujitsu Lifebook E654 with 2.2 Ghz Intel Core i7 MQ CPU with 16 GB ram.

6.1 Capital budgeting

A capital budgeting problem faced by companies engaged in procurement of R&D contracts consists of optimally allocating limited funds to the suggested projects, see Petersen [1967]. The problem can be formulated as a binary ILP $\max\{cx \mid Ax \leq b, x \in \{0, 1\}^n\}$, where $x_j = 1$ if project j is selected and $x_j = 0$ otherwise, $j = 1, \dots, n$. Moreover, c is the vector of contract volumes, A is a cost matrix, and b is the vector of budgets.

In line with Dawande and Hooker [2000], we solve the largest of the capital budgeting instances proposed by Petersen [1967]. This instance includes 50 projects and 5 budget constraints.

Table 5: Sensitivity intervals for each objective function coefficient of the capital budgeting problem. All entries are in thousands of dollars.

j	c_j	Δc_j	j	c_j	Δc_j	j	c_j	Δc_j
1	560	$[-\infty, 43]$	18	115	$[-\infty, 41]$	35	71	$[-13, +\infty]$
2	1125	$[-\infty, 189]$	19	82	$[-44, +\infty]$	36	49	$[-19, +\infty]$
3	300	$[-\infty, 62]$	20	22	$[-13, +\infty]$	37	108	$[-58, +\infty]$
4	620	$[-74, +\infty]$	21	631	$[-\infty, 72]$	38	116	$[-18, +\infty]$
5	2100	$[-\infty, 809]$	22	132	$[-\infty, 57]$	39	90	$[-64, +\infty]$
6	431	$[-140, +\infty]$	23	420	$[-76, +\infty]$	40	738	$[-382, +\infty]$
7	68	$[-\infty, 18]$	24	86	$[-\infty, 39]$	41	1811	$[-884, +\infty]$
8	328	$[-85, +\infty]$	25	42	$[-13, +\infty]$	42	430	$[-286, +\infty]$
9	47	$[-28, +\infty]$	26	103	$[-19, +\infty]$	43	3060	$[-2076, +\infty]$
10	122	$[-\infty, 33]$	27	215	$[-75, +\infty]$	44	215	$[-38, +\infty]$
11	322	$[-87, +\infty]$	28	81	$[-29, +\infty]$	45	58	$[-\infty, 13]$
12	196	$[-19, +\infty]$	29	91	$[-71, +\infty]$	46	296	$[-\infty, 77]$
13	41	$[-32, +\infty]$	30	26	$[-\infty, 13]$	47	620	$[-181, +\infty]$
14	25	$[-\infty, 14]$	31	49	$[-43, +\infty]$	48	418	$[-191, +\infty]$
15	425	$[-248, +\infty]$	32	420	$[-285, +\infty]$	49	47	$[-18, +\infty]$
16	4260	$[-1389, +\infty]$	33	316	$[-\infty, 18]$	50	81	$[-61, +\infty]$
17	416	$[-245, +\infty]$	34	72	$[-31, +\infty]$			

Table 5 shows the results of a change to a single objective function coefficient. For example, the contract volume of project 1 is \$560,000. Since $x_1^* = 0$, the current allocation of funds remains optimal for any decrease in volume. Moreover, the allocation is optimal as long as the volume is less than or equal to \$603,000, i.e., the volume can increase by at most 8%. This also means that if the contract volume of project 1 increases by at least $\Delta_1 = \$43,000$, it will be optimal to invest in project 1. Similarly, the contract volume of project 4 is \$620,000 and $x_4^* = 1$. The volume can increase indefinitely but decrease by at most $\Delta_4 = \$74,000$ for the current allocation to remain optimal. If the contract volume of project 4 falls below \$546,000 the companies should not invest in project 4. An interesting question to ask is what happens if the contract volumes of project 1 and project 2 vary simultaneously? Using Theorem 3 it can be shown that the optimal solution does not change if $\Delta_1 \leq \$43,000$, $\Delta_4 \geq -\$135,000$ and $\Delta_1 - \Delta_4 \leq \$74,000$. We see that if the contract volume of project 1 increases with, say, $\Delta_1 = \$40,000$, then the optimal solution remains optimal if the contract volume of project 4 decreases at most $\Delta_4 = -\$34,000$. Now, what happens if the contract volumes of projects 1, 2 and 4 vary simultaneously? By Theorem 5, the optimal solution does not change if $\Delta_1 \leq \$43,000$, $\Delta_2 \leq \$189,000$ and $\Delta_1 - \Delta_4 \leq \$74,000$.

To determine the sensitivity intervals for changes in the contract volumes of x_j with $j \in \{1, 2, \dots, 50\}$, the total CPU time is 7.6 seconds, with an average time of only 0.15 seconds. For simultaneous changes in the contract volumes of the variables x_i, x_j with $i, j \in \{1, 2, \dots, 50\}$, the total CPU time for all 1,225 sensitivity regions is 400 seconds, i.e. 6-7 minutes, with an average time of 0.33 seconds. Finally, for the variables x_i, x_j, x_k with $i, j, k \in \{1, 2, \dots, 20\}$, the total and average CPU time for the 1,140 sensitivity regions is 820 seconds and 0.72 seconds, respectively. The results demonstrate that for binary integer linear programs, a complete sensitivity analysis of single variables or pairs of variables can be carried out within short computation time. Furthermore, it is feasible to obtain the sensitivity regions for selected sets of more than two variables.

6.2 Production lot sizing

The production lot sizing problem determines an optimal production plan for various types of items over a finite planning horizon. The aim is to cover demand of all time periods, while respecting capacity constraints (on the time spent on production and setup) and allowing for inventory and backloging of production, and at the minimal inventory, backlog and setup costs. The problem is formulated as a MILP.

We use an extended version of the lot sizing problem, Model 1 by Molina, Morabito, and Alexandre de Araujo [2016], which includes piecewise linear and convex costs of transporting the items using pallets, assuming distinct types of items cannot be mixed on the same pallet, see (8). Parameters and variables are defined in Appendix C.

$$\begin{aligned}
\min \quad & \sum_{i=1}^n \sum_{t=1}^T (h_{it}^+ I_{it}^+ + h_{it}^- I_{it}^- + s_{it} y_{it}) + \sum_{t=1}^T (c_0 + c_1 \sum_{i=1}^n a_{it} + c_2 \sum_{i=1}^n b_{it}) \\
\text{s.t.} \quad & I_{i,t-1}^+ - I_{i,t-1}^- + x_{it} - I_{i,t}^+ + I_{i,1}^- = d_{it}, \quad i = 1, 2, \dots, n, t = 1, 2, \dots, T \\
& x_{it} - M y_{it} \leq 0, \quad i = 1, 2, \dots, n, t = 1, 2, \dots, T \\
& \sum_{i=1}^n (p_i x_{it} + q_i y_{it}) \leq Cap_t, \quad t = 1, 2, \dots, T \\
& x_{it} - m_i (a_{it} + b_{it}) \leq 0, \quad i = 1, 2, \dots, n, t = 1, 2, \dots, T \\
& \sum_{i=1}^n a_{it} \leq R, \quad t = 1, 2, \dots, T \\
& I_{i0}^+ = I_{i0}^- = I_{iT}^+ = I_{iT}^- = 0, \quad i = 1, 2, \dots, n \\
& x_{it} \geq 0, I_{it}^+ \geq 0, I_{it}^- \geq 0 \quad i = 1, 2, \dots, n, t = 1, 2, \dots, T \\
& a_{it}, b_{it} \in \mathbb{Z}_+, y_{it} \in \{0, 1\} \quad i = 1, 2, \dots, n, t = 1, 2, \dots, T.
\end{aligned}$$

We illustrate our method on an instance with $(n, T) = (3, 8)$. We generate data as described for Model 1 by Molina et al. [2016] and include the data in Appendix C.1. This instance contains 72 continuous variables, 48 integer variables and 24 binary variables and 100 constraints. A number of interesting questions arises:

- The inventory cost for product 2 in period 2 is $h_{2,2}^+ = \$3$. How much can this cost vary while the solution remains optimal? Using the results of Section 4, any change should be in the interval $[-3; 0]$ (assuming that the inventory cost is nonnegative). Hence, if the inventory cost increases, the optimal production plan changes. The CPU time to obtain the answer is 3.64 seconds.

- The setup cost for product 2 in period 2 is $s_{2,2} = \$50$. How much can this cost vary? Our analysis shows that the cost can increase to as much as \$8280, suggesting that setup costs have limited impact on the optimal production plan. The CPU time is only 0.69 seconds.
- The objective function coefficient of $a_{2,6}$ is \$50. In which range does the current production plan remain optimal? The answer is that the change should be in the interval $[-50; 32]$ (assuming that the coefficient is nonnegative; otherwise, the lower bound is -\$200). The CPU is 4.38 seconds.
- The objective function coefficient of $b_{2,6}$ is \$200 and the sensitivity interval is $[-32; \infty]$. Thus, if the higher unit cost of product 2 in period 6 is at least \$168, it remains optimal to transport only at the lower unit cost. The CPU time to obtain the answer is 3.41 seconds.
- For which simultaneous changes Δ_1, Δ_2 to the objective function coefficients of the backlog and setup costs of product 2 in period 2, respectively, does the optimal solution remain the same? Using Theorem 3, it can be shown that any changes should satisfy the constraints $\Delta_1 \geq -10$, $\Delta_2 \geq -50$, $-148\Delta_1 + \Delta_2 \leq 8230$, $-240\Delta_1 + \Delta_2 \leq 8345$, $-314\Delta_1 + \Delta_2 \leq 8345$, $-474\Delta_1 + \Delta_2 \leq 8649$, $-571\Delta_1 + \Delta_2 \leq 8672$, $-590\Delta_1 + \Delta_2 \leq 8805$, $-667\Delta_1 + \Delta_2 \leq 9544$. The CPU time is 423.97 seconds, indicating that sensitivity analysis for more than a single coefficient is significantly more time consuming than sensitivity analysis for a single coefficient.

We proceed to further investigate the CPU times for changes to single objective function coefficients. Tables 6 and 7 are based on the averages of 10 instances with $(n, T) = (3, 8)$ and $(n, T) = (3, 12)$, respectively. Both tables report the times for computing the sensitivity intervals for all variables and for five separate groups of variables, e.g. the group $I_{it}^+, i = 1, \dots, n, t = 1, \dots, T$ is denoted by I^+ . The left-hand-sides of the tables show statistics for computing the sensitivity interval for a single variable in the group and the right-hand-sides for all variables in the group.

Table 6: CPU-times (seconds) for computing the sensitivity intervals for the objective function coefficients for the five groups of variables with $n = 3$ and $T = 8$. Each group contains 24 variables.

Variables	Single variable					All variables				
	Average	Std.dev.	Median	Min	Max	Average	Std.dev.	Median	Min	Max
I^+	4.39	12.95	1.56	0.42	161.25	105.40	139.07	32.86	21.74	438.37
I^-	2.77	12.23	1.16	0.41	187.48	66.37	77.71	26.40	16.08	250.06
y	0.92	2.26	0.31	0.13	24.27	22.13	27.96	7.84	4.81	78.25
a	2.00	1.75	1.32	0.44	12.84	48.03	34.12	31.42	20.75	112.00
b	1.52	1.11	0.94	0.41	5.11	36.40	25.62	22.53	14.55	86.38
All	2.32	8.16	1.03	0.13	187.48	278.06	293.11	120.26	80.73	907.83

Table 6 shows that on average it takes 2.32 seconds to compute a sensitivity interval, with a minimum of 0.13 seconds, a maximum of 3 minutes and 7 seconds (187.48 seconds) and a standard deviation of 8.16 seconds. It should be remarked that a few hard instances makes the average high. In fact, the median is only around a second. Evidently, the average time to compute all sensitivity intervals is proportional to the number of variables, i.e. 4 minutes and 38 seconds (278.06 seconds). The same applies to the median, which is 2 minutes (120.26 seconds). However, both the span between the minimum and maximum and the standard deviation increases much less than proportionally, i.e. 1-15 minutes (80.73-907.83 seconds) and 5 minutes (293.11 seconds), respectively. As expected from Section 4.3, binary variables are less time consuming than others. The non-binary variables

Table 7: CPU-times (seconds) for computing the sensitivity intervals for the objective function coefficients for the five groups of variables with $n = 3$ and $T = 12$. Each group contains 36 variables.

Variables	Single variable					All variables				
	Average	Std.dev.	Median	Min	Max	Average	Std.dev.	Median	Min	Max
I^+	12.48	39.00	4.32	0.61	661.20	449.15	499.17	259.29	49.18	1417.63
I^-	6.89	14.10	3.20	0.63	215.48	247.99	288.60	126.77	34.85	860.83
y	1.96	7.73	0.80	0.16	138.66	70.60	83.24	44.38	14.52	294.11
a	6.28	7.58	3.28	0.63	40.69	226.20	256.53	148.37	42.37	880.45
b	5.39	7.33	2.18	0.64	49.39	194.06	248.41	105.96	33.02	846.77
All	6.60	19.72	2.15	0.16	661.20	1179.58	1316.76	671.26	172.01	4272.51

have comparable medians, whereas the computation times for the continuous inventory variables vary significantly more than for the number of pallets. Similar conclusions hold when increasing the size of the instances, see Table 7. Compared to Table 6 the median time to compute a sensitivity interval doubles whereas the average roughly increases by a factor three, and the increases are even higher for the time to compute all sensitivity intervals.

7 Conclusions

This paper presents a multi-objective optimization approach to sensitivity analysis of the objective function coefficients in MILP. For variations in a single objective function coefficient, we show how to determine the maximal sensitivity region by bi-objective mixed-integer linear programming. We show that it suffices to determine the two extreme nondominated points in objective space that are adjacent to the optimal solution. We extend the methodology for simultaneous changes to two or more coefficients, determining the sensitivity region by use of multi-objective analysis.

Our approach can be used with any solution method for multi-objective optimization. For bi-objective mixed-integer linear programming, the sets of extreme nondominated points can be found using the first phase of the so-called two-phase method, see for example Przybylski et al. [2008]. As shown, it is sufficient to find the two adjacent points. The first phase of the two-phase method can be easily modified to do so. A number of recent papers describe how to determine the set of nondominated points of a general MO-MILP. Özpeynirci and Köksalan [2010] present a maximal algorithm for finding all extreme nondominated points of a MO-MILP, Przybylski, Gandibleux, and Ehrgott [2010] finds all nondominated extreme points for a multi-objective integer program, and Kirlik and Sayin [2014] determine all nondominated points to multi-objective discrete models. One or more of these algorithms may be used. Available software packages can likewise find the set of nondominated points of a MO-MILP, see for instance Gandibleux, Soleilhac, Przybylski, Lucas, Ruzika, and Halffmann [2017a] and Gandibleux, Soleilhac, Przybylski, and Ruzika [2017b].

Although computational performance remains to be further tested, existing methods for multi-objective optimization demonstrate the potential of our procedure. That is, as multi-objective solvers become more efficient, they can be called from a MILP solver and used to find sensitivity regions. This may both be *a priori* or *a posteriori*, i.e. after the optimal MILP solution has been observed.

Our method does not immediately generalize to changes to the right-hand-side and the coefficients of the constraint matrix in MILP. We therefore leave these extensions to future research.

Acknowledgemet

Dedicated to the memory of Kim Allan Andersen. The first author of this paper, Professor Kim Allan Andersen, died unexpectedly in January 2021 before submitting this paper. Kim was a huge capacity for OR in Denmark and will be missed. We would like to thank the anonymous referees for improving the paper.

Appendices

A Proofs of Lemma 3 and Proposition 1

Proof of Lemma 3

Proof. Lemma 2 shows that $lb(\mathcal{X}) \leq lb^+$ and $ub^+ \leq ub(\mathcal{X})$. If $c_k > 0$ and $lb(\mathcal{X}_E^+) < -c_k$ then $lb^+ = -c_k$, and if $c_k < 0$ and $ub(\mathcal{X}_E^+) > -c_k$ then $ub^+ = -c_k$. Therefore, we have that $lb(\mathcal{X}) \leq -c_k = ub^-$ for $c_k > 0$ and $lb^- = -c_k \leq ub(\mathcal{X})$ for $c_k < 0$.

Consider $c_k > 0$ and $\mathcal{X}_E^- \neq \emptyset$. Using Lemma 1, it remains to show that $lb(\mathcal{X}) \leq lb^-$. Assume contrarily that $\exists \Delta : lb^- < \Delta < lb(\mathcal{X})$. As $\Delta \notin [lb(\mathcal{X}), ub(\mathcal{X})]$, we have that $\exists(\hat{x}, \hat{y}) \in \mathcal{X} : \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$. Since, however, $\Delta \in [lb^-, lb(\mathcal{X})] \subseteq [lb(\mathcal{X}_E^-), ub(\mathcal{X}_E^-)]$, we have $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*)$, $(x, y) \in \mathcal{X}_E^-$. Hence, $(\hat{x}, \hat{y}) \notin \mathcal{X}_E^-$. This implies that there exists $(\bar{x}, \bar{y}) \in \mathcal{X}_E^-$, such that (i) $-c_k \bar{x}_k \geq -c_k \hat{x}_k$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i \hat{x}_i + h \hat{y}$. We multiply (i) by $(c_k + \Delta)/c_k \leq 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) \geq \nu_\Delta(\hat{x}, \hat{y})$. But then $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, contradicting $\Delta \in [lb(\mathcal{X}_E^-), ub(\mathcal{X}_E^-)]$.

For $c_k > 0$ and $\mathcal{X}_E^- = \emptyset$, we aim to show that $lb(\mathcal{X}) \geq -c_k$. Assume contrarily that $\exists \Delta : lb(\mathcal{X}) < \Delta < -c_k$. As $\Delta \in [lb(\mathcal{X}), ub(\mathcal{X})]$, we have that $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*)$, $(x, y) \in \mathcal{X}$. Since, however, $\mathcal{X}_E^- = \emptyset$, there exists $(\bar{x}, \bar{y}) \in \mathcal{X}$, such that (i) $-c_k \bar{x}_k \geq -c_k x_k^*$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i x_i^* + h y^*$, with at least one of the inequalities being strict. We multiply (i) by $(c_k + \Delta)/c_k < 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, which is a contradiction.

Consider $c_k < 0$ and $\mathcal{X}_E^- \neq \emptyset$. Using Lemma 1, it remains to show that $ub^- \leq ub(\mathcal{X})$. Assume contrarily that $\exists \Delta : ub^- > \Delta > ub(\mathcal{X})$ and proceed as in the case of $c_k > 0$.

The case of $c_k < 0$ and $\mathcal{X}_E^- = \emptyset$ is similar to $c_k > 0$. □

Proof of Proposition 1

Proof. First, consider the upper bound. Assume that $\mathcal{X}_E^+ \neq \emptyset$. To see that $ub(\mathcal{X}_E^+) = ub(\mathcal{X})$, assume contrarily that $\exists \Delta : ub(\mathcal{X}) < \Delta < ub(\mathcal{X}_E^+)$. As in the proof of Lemma 2, $\exists(\hat{x}, \hat{y}) \in \mathcal{X} \setminus \mathcal{X}_E^+ : \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}_E^+$ such that (i) $\bar{x}_k \geq \hat{x}_k$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i \hat{x}_i + h \hat{y}$. We multiply (i) by $\Delta \geq 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) \geq \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$, which is a contradiction.

For $\mathcal{X}_E^+ = \emptyset$, we aim to show that $ub(\mathcal{X}) = 0$. Assume contrarily that $\exists \Delta : \Delta > 0$. Now, $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*)$, $(x, y) \in \mathcal{X}$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}$, such that (i) $\bar{x}_k \geq x_k^*$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h \bar{y} \geq \sum_{i \neq k} c_i x_i^* + h y^*$, with at least one of the inequalities being strict. We multiply (i) by $\Delta > 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, which is a contradiction.

Next, consider the lower bound. Assume that $\mathcal{X}_E^- \neq \emptyset$. As before, to see that $lb(\mathcal{X}) = lb(\mathcal{X}_E^-)$, assume that $\exists \Delta : lb(\mathcal{X}_E^-) < \Delta < lb(\mathcal{X})$. As in the proof of Lemma 3, $\exists(\hat{x}, \hat{y}) \in \mathcal{X} \setminus \mathcal{X}_E^- : \nu_\Delta(\hat{x}, \hat{y}) >$

$\nu_\Delta(x^*, y^*)$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}_E^-$ such that (i) $-\bar{x}_k \geq -\hat{x}_k$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h\bar{y} \geq \sum_{i \neq k} c_i \hat{x}_i + h\hat{y}$. We multiply (i) by $\Delta \leq 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) \geq \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$, which is again a contradiction.

For $\mathcal{X}_E^- = \emptyset$, we aim to show that $lb(\mathcal{X}) = 0$. Assume contrarily that $\exists \Delta : \Delta < 0$. Now, $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*)$, $(x, y) \in \mathcal{X}$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}$, such that (i) $-\bar{x}_k \geq -x_k^*$ and (ii) $\sum_{i \neq k} c_i \bar{x}_i + h\bar{y} \geq \sum_{i \neq k} c_i x_i^* + h y^*$, with at least one of the inequalities being strict. We multiply (i) by $\Delta < 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, which is a contradiction. \square

B Variations of three or more objective function coefficients

For $K \subseteq \{1, 2, \dots, n + p\}$, we will determine the sensitivity region for simultaneous changes $(\Delta_k : k \in K)$ to the objective function coefficients c_k , $k \in K$ of the vector c such that (x^*, y^*) remains optimal to Π . The same analysis applies to changes in any $|K|$ coefficients of the vectors c and h .

For a subset $\hat{\mathcal{X}}$ of \mathcal{X} , define the region

$$\Omega(\hat{\mathcal{X}}) = \left\{ (\Delta_k : k \in K) \in \mathbb{R}^{|K|} \mid \nu_{(\Delta_k : k \in K)}(x, y) \leq \nu_{(\Delta_k : k \in K)}(x^*, y^*), (x, y) \in \hat{\mathcal{X}} \right\}.$$

If $\hat{\mathcal{X}} \neq \emptyset$, then $\Omega(\hat{\mathcal{X}})$ is a convex set. If $\hat{\mathcal{X}} = \emptyset$, we define $\Omega(\hat{\mathcal{X}}) = \emptyset$.

Let $\Theta = \{\theta \in \mathbb{R}^{|K|} \mid \theta_k \in \{+, -\}, k \in K\}$ and $\theta \in \Theta$. Define the multi-objective problems Π_{MO}^θ with the $|K| + 1$ objective functions

$$z^\theta(x, y) = (\text{sgn}(\theta_1)c_1x_1, \dots, \text{sgn}(\theta_{|K|})c_{|K|}x_{|K|}, \sum_{\substack{i=1 \\ i \notin K}}^n c_i x_i + hy),$$

and feasible set \mathcal{X} . Denote by \mathcal{X}_E^θ the efficient set to Π_{MO}^θ and define

$$\Omega^\theta = \{(\Delta_k : k \in K) \in \Omega(\mathcal{X}_E^\theta) \mid \text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(\theta_k) \cdot \text{sgn}(c_k), 0\}, k \in K\}.$$

We can now formulate the general result.

Theorem 5. *The sensitivity region is $\cup_{\theta \in \Theta} \Omega^\theta$.*

Proof. The proof is similar to the proof of Theorem 3 and is therefore omitted. \square

Let $\theta \in \Theta$. Similarly to Corollary 1 and Corollary 2 it suffices to consider the set Ω^θ of the extreme nondominated points in objective space. Consider the image set

$$\mathcal{Z}_N^\theta = \{(c_1x_1, \dots, c_{|K|}x_{|K|}, \sum_{\substack{i=1 \\ i \notin K}}^n c_i x_i + hy) \mid (x, y) \in \mathcal{X}_E^\theta\}.$$

and let $\mathcal{Z}_{N,e}^\theta$ denote the set of extreme points of the set

$$\mathcal{Z}_{\leq}^\theta = \text{conv} \left(\mathcal{Z}_N^\theta + \{z \in \mathbb{R}^q : z \leq 0\} \right).$$

Then,

$$\Omega^\theta = \{(\Delta_k : k \in K) \in \mathbb{R}^{|K|} \mid \sum_{k \in K} \frac{\Delta_k}{c_k} (z_k - z_k^*) \leq \sum_{k \in K} (z_k^* - z_k) + z_{|K|+1}^* - z_{|K|+1}, \\ (z_1, \dots, z_{|K|}, z_{|K|+1}) \in \mathcal{Z}_{N,e}^\theta, \text{sgn}(c_k + \Delta_k) \in \{\text{sgn}(\theta_k) \cdot \text{sgn}(c_k), 0\}, k \in K\}.$$

C Production lot sizing: Nomenclature

Parameters:

$i = 1, 2, \dots, n$	Number of types of items
$t = 1, 2, \dots, T$	Number of periods in the planning horizon
s_{it}	Setup cost for the production of item i in period t
h_{it}^+	Unit inventory cost of item i in period t
h_{it}^-	Penalty for delay of one unit of item i in period t
d_{it}	Demand for item i in period t
p_i	Time required to produce a unit of item i
q_i	Setup time for the production of item i
Cap_t	Production capacity in period t
M	A sufficiently large positive number. We use $M = \sum_{i=1}^n \sum_{t=1}^T d_{it}$
m_i	Number of items of type i that can be placed on the same pallet
c_0	Fixed monthly cost of the contract
c_1	Unit transport cost of first R pallets used
c_2	Unit transport cost of the other pallets ($c_1 < c_2$)
R	The contracted number of hired pallets with cost c_1

Variables:

x_{it}	Amount to be produced of item i in period t
I_{it}^+	Inventory of item i in period t
I_{it}^-	Backlog of item i in period t
y_{it}	Binary variable indicating the production of item i in period t ($y_{it} = 1$ if $x_{it} > 0$; $y_{it} = 0$ otherwise)
a_{it}	Number of pallets transported containing item i in period t with unit cost c_1
b_{it}	Number of pallets transported containing item i in period t with unit cost c_2

C.1 Production lot sizing: Data

Below, we provide the data for instance 1 with $(n, T) = (3, 8)$. All data are generated as described for Model 1 in Molina et al. [2016].

$$(s_{it}, h_{it}^+, h_{it}^-) = (50, 3, 10), \quad i = 1, \dots, 3, \quad t = 1, \dots, 8,$$

	d_{it}							
i/t	1	2	3	4	5	6	7	8
1	208	117	180	177	241	159	127	186
2	527	917	728	349	759	382	730	330
3	264	329	353	262	297	232	199	337

$$(p_1, p_2, p_3) = (1, 1, 1),$$

$$(q_1, q_2, q_3) = (29, 28, 30),$$

$$Cap_t = 1336, \quad t = 1, \dots, 8,$$

$$M = 8390,$$

$$(m_1, m_2, m_3) = (131, 81, 74),$$

$$(c_0, c_1, c_2) = (0, 50, 200),$$

$R = 9$.

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