The facets of the set packing polytope: A logical interpretation

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Abstract

In this paper we present a logical interpretation of all the facets of the the set packing polytope. The approach is based on results obtained in probabilistic logic (probabilistic satisfiability) and reveals an interesting connection between probabilistic logic and integer linear programming.

Keywords: Integer programming, probabilistic logic, set packing polytope.

1 Introduction

The aim of this paper is to give a logical interpretation of all the facets of the set packing polytope. Our interpretation explains why all the particular facets are necessary in the description of the set packing polytope. To the best of our knowledge such an explanation has not been given before. Most of the litterature on the set packing polytope has concentrated on deriving facets of the polytope in an algebraic way, see for example Balas and Zemel [7] and Padberg [13], but none of these papers gives a logical explanation on why the facets are necessary in the description of the set packing polytope.

There are examples in integer linear programming where it is possible to give an explicit description of all the facets of the convex hull of the feasible set (i.e. the convex hull of a set of integral points), and furthermore a logical interpretation of the facets can be given. Probably the best known example is the matching polytope which we will briefly review. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple undirected graph with nodeset $\mathcal{V} = \{1, 2, \ldots, n\}$ and edgeset $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. A matching $\mathcal{M} \subseteq \mathcal{E}$ is a subset of edges such that each node in the subgraph $\mathcal{G}(\mathcal{M}) = (\mathcal{V}, \mathcal{M})$ is met by at most one edge. We need some more notation:

- Let $v \in \mathcal{V}$. Define $\delta(\{v\}) = \{e \in \mathcal{E} | e \text{ is incident to } v\}$.
- Let $U \subseteq \mathcal{V}$. Define $\mathcal{E}(U) = \{e \in \mathcal{E} | \text{both ends of } e \text{ are in } U\}$

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- Let $U \subseteq \mathcal{V}$. Then U is an odd set of vertices if the cardinality $|U|$ of U is odd and at least 3.
- B^m is the set of binary m-vectors.

The matching polytope is defined as the convex hull of the constraints (1):

$$\begin{array}{l} \sum_{e \in \delta(\{v\})} x_e \leq 1, \quad \forall v \in \mathcal{V} \\ x \in B^m. \end{array} \tag{1}$$

It is well known that the convex hull of the constraints (1) is given by the constraints (2):

$$\begin{array}{l} \sum_{e \in \delta(\{v\})} x_e \leq 1, \quad \forall v \in \mathcal{V} \\ \sum_{e \in \mathcal{E}(U)} x_e \leq (|U| - 1)/2, \;\text{ for all odd sets }\; U \subseteq \mathcal{V} \\ x \in \mathbb{R}_+^m. \end{array} \tag{2}$$

We now see that it is possible to give a logical explanation of all the facets of the matching polytope. The first set of constraints expresses the fact that in a matching $\mathcal{M}$ each node is met by at most one edge whereas the second set of constraints expresses the fact that a matching $\mathcal{M}$ can have at most $(|U| - 1)/2$ edges with both ends inside an odd set of vertices U.

With the matching poytope as a motivating example we will show how to give a logical interpretation of all the facets of the set packing polytope. It turns out that the theory of probabilistic logic provide us with the tools to do so.

Probabilistic logic originally dates 150 years back to Boole [10] but was reinvented by Nilsson [12]. In the past 15 years probabilistic logic has been an interesting research topic, and many new results has been obtained, see Andersen [1, 2], Andersen and Hooker [3, 4, 5], Andersen and Pretolani [6], Chandru and Hooker [8], Georgakopoulus *et al.* [9], and Hansen *et al.* [11]. A short review of probabilistic logic is given in section 2.

The results obtained in Andersen [1] are the ones used in this paper to obtain a logical interpretation of the facets for the set packing polytope. For any digraph the paper describes a set of logical sentences which can be represented by that particular digraph. It is assumed that each sentence is true with some fixed probability (which might be one). The probabilistic satisfiability problem (PSAT) is to determine whether or not the assignment of probabilities to the sentences is consistent. In the paper a complete characterization of the probabilities which can consistently be assigned to the sentences is given in terms of a set covering polytope. This set covering polytope is defined as the convex hull of binary solutions to a set of linear constraints with coefficients 0 and 1 and with exactly two entries different from 0 in each row. This is the key observation to obtain a logical interpretation of the facets of the set packing polytope. This will be explained in detail in section 4.

The logical interpretation of the facets of the set packing polytope will follow the lines:

- Given any set packing polytope it is possible to associate a set of logical sentences with it.
- Each point in the set packing polytope represents an assignment of probabilities which can consistently be assigned to the logical sentences. Any point outside the set packing polytope is not a consistent assignment of probabilities to the sentences.
- Therefore, each facet of the set packing polytope can be given the logical interpretation that it gives a necessary condition on the probabilities which can consistently be assigned to the sentences. The set of facets of the set packing polytope describes the necessary and sufficient conditions that a consistent assignment of probabilities to the sentences should fulfil.

The results obtained in this paper reveals a surprising and interesting connection between integer linear programming and probabilistic logic. Integer linear programming typically concerns the description of the convex hull of integral points satisfying a set of linear constraints wheras probabilistic logic (probabilistic satisfiability) aims to describe the set of probabilities which can consistently be assigned to some set of logical sentences. It is very interesting to note that two seemingly distinct disciplines such as integer linear programming and probabilistic satisfiability does in fact have something to contribute to each other.

The remaining parts of the paper is organized as follows. In section 2 we give a short introduction to consistency in probabilistic logic. In section 3 we review some results obtained in Andersen [1] on logical sentences represented by digraphs. The section contains a complete characterization of the probabilistic satisfiability problem (PSAT) for sentences represented by digraphs. Section 4 shows how to give a logical interpretation of all the facets of a set packing problem. Finally section 5 contains the conclusions.

2 Consistency in probabilistic logic

In this section we give a short introduction to consistency in probabilistic logic. For a comprehensive introduction to probabilistic logic we refer to Chandru and Hooker [8].

The problem of consistency in probabilistic logic, also called probabilistic satisfiability or PSAT for short, is defined as follows:

> Let $S = \{S_1, \ldots, S_m\}$ be a set of m logical sentences defined on a set of n propositional variables $\mathcal{X} = \{x_1, \ldots, x_n\}$ with the usual connectives $\vee$ (disjunction), $\wedge$ (conjunction), and $\neg$ (negation). Denote by $\pi = (\pi_1, \ldots, \pi_m\}$ a probability vector such that $\pi_i = \Pr(S_i \text{ is true})$, $i = 1, \ldots, m$, where Pr is short for probability. Determine whether or not this assignment of probabilities to the sentences is a consistent assignment.

To answer this question we need to understand what is meant with the probability of a logical sentence. Let a *possible world* be an assignment of values *true* or *false* to the n propositional variables. There are $N = 2^n$ possible worlds. Let $p = (p_1, \ldots, p_N)^t$ be a probability distribution over the set of possible worlds. Then we say that the probability of a logical sentence is the sum of the probabilities of the possible worlds in which the sentence is true.

Let A be an $m \times N$ matrix such that $a_{ij} = 1$ if sentence S_i is true in possible world j, and $a_{ij} = 0$, otherwise. Let e be an N-vector of ones. Then PSAT asks if there is a probability distribution p such that the system (3) has a solution:

$$\begin{aligned} Ap &= \pi \\ e^t p &= 1 \\ p &\geq 0. \end{aligned} \tag{3}$$

Notice that π is a consistent assignment of probabilities to the logical sentences in question if and only if π can be written as convex combination of the columns of A. Clearly, it is sufficient to consider the *distinct* columns of A.

Example 1

Suppose we have four logical sentences S_1, S_2, S_3 and S_4 defined on four propositional variables x_1, x_2, x_3 and x_4:

$$\begin{aligned} S_1 &: \quad x_1 \\ S_2 &: \quad x_1 \rightarrow x_2 \\ S_3 &: \quad (x_1 \wedge x_2) \rightarrow x_3 \\ S_4 &: \quad (x_2 \wedge x_3) \rightarrow x_4. \end{aligned}$$

For convenience we have used the connective $\rightarrow$ (implies). This connective can be expressed using the usual connectives as follows: $S_1 \rightarrow S_2 \Longleftrightarrow \neg S_1 \vee S_2$.

Suppose the possible worlds are ordered $(x_1, x_2, x_3, x_4) = (0,0,0,0), (x_1, x_2, x_3, x_4) = (0,0,0,1), \ldots, (x_1, x_2, x_3, x_4) = (1,1,1,1)$. Then A is given by:

$$A = \left\{ \begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right\}$$

Assign probabilities $\pi = (\pi_1,\ \pi_2,\ \pi_3,\ \pi_4)$ to the four logical sentences, where $\pi_i = \Pr(S_i$ is true$)$, $i = 1, \ldots, 4$. Then π is a consistent assignment of probabilities to the four logical sentences if and only if there is a probability distribution $p = (p_1, \ldots, p_{16})^t$, such that $Ap = \pi$ has a solution.

□

Our intention is to find a set of necessary and sufficient conditions on the probability vector π which ensures that PSAT has an affirmative answer. To do that PSAT is expressed in a slightly different way:

$$\begin{array}{ll} \min & 0 \cdot p \\ \text{s.t.} & \\ & \begin{array}{rcl} Ap & = & \pi \\ e^t p & = & 1 \\ p & \geq & 0. \end{array} \end{array} \tag{4}$$

Then PSAT has a positive answer if and only if the optimal value of (4) is equal to zero. The dual of (4) is:

$$\begin{array}{ll} \max & y_0 + y \cdot \pi \\ \text{s.t.} & \\ & e \cdot y_0 + yA \leq 0. \end{array} \tag{5}$$

This leads to the following theorem

Theorem 1 *([11]) PSAT has an affirmative answer if and only if the inequality* $(1, \pi)^t \cdot (y_0, y) \leq 0$ *holds for all extreme rays* (y_0, y) *of (5).*

3 Logical sentences represented by digraphs

In this section we review some of the results in Andersen [1]. Let $\mathcal{D} = (\mathcal{V}, \mathcal{A})$ be a directed graph with node-set $\mathcal{V} = \{1, \ldots, n\}$ and arc-set $\mathcal{A}$. This digraph represents n logical sentences defined on n propositional variables $x_1, x_2, \ldots, x_n$ as follows:

- If a node, say node j, has no incoming arcs then it represents the logical sentence S_j defined by: $S_j \equiv x_j$.

  Let $y_j = v(S_j)$ denote the truth value of the sentence S_j.

- If node r has several incoming arcs, for instance arcs from nodes $i_1, i_2, \ldots, i_t$, then node r represents the logical sentence S_r defined by: $S_r \equiv (x_{i_1} \wedge x_{i_2} \ldots \wedge x_{i_t}) \rightarrow x_r$.

  Let $y_r = v(S_r)$ denote the truth value of the sentence S_r.

Example 2

Consider the directed graph below

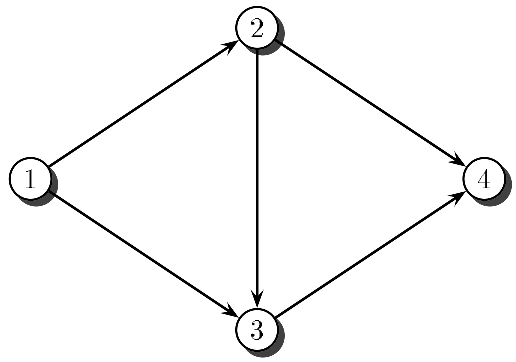

Figure 1.

We associate propositional variables x_1, x_2, x_3 and x_4 with the four nodes in the digraph. The digraph represents the set of logical sentences:

$$\begin{aligned} S_1: &\quad x_1 \\ S_2: &\quad x_1 \rightarrow x_2 \\ S_3: &\quad (x_1 \wedge x_2) \rightarrow x_3 \\ S_4: &\quad (x_2 \wedge x_3) \rightarrow x_4. \end{aligned}$$

which happens to be exactly the sentences used in example 1.

Notice, that given a possible world (x_1, x_2, x_3, x_4) the corresponding column in A is the truthvalues of the four logical sentences. So what we are looking for is a set of probabilityvectors π such that π can be written as a convex combination of the columns of A, i.e. a convex combination of truthvalues of the four logical sentences.

To each of the four nodes we assign a y-variable. The y's have the interpretation that they are the truth values of the logical sentences corresponding to the four nodes, i.e.

$$y_1 = v(S_1), \quad y_2 = v(S_2), \quad y_3 = v(S_3), \quad y_4 = v(S_4).$$

In this tiny example it turns out that the *distinct* columns of A are exactly the set of $0-1$ solutions to the following set of inequalities:

$$\begin{aligned} y_1 + y_2 &\geq 1 \\ y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 1 \\ y_2 + y_4 &\geq 1 \\ y_3 + y_4 &\geq 1. \end{aligned}$$

These equations can be rewritten as: $\{y \in B^4 \mid Ey \geq 1\}$ where E is the arc-node incidence matrix for the underlying (undirected) graph of the digraph, B^4 is the set of 4-dimensional $\{0,1\}$- vectors and 1 is a column vector of 1's of appropriate dimension.

□

In example 1 the set of distinct columns of A could be characterized using the arc-node incidence matrix for the underlying (undirected) graph of the digraph. This property holds true in general.

Theorem 2 *([1]) Suppose we have a set of logical formulas represented by a digraph. Then the distinct columns of A is exactly the set of solutions to the system $\{y \in B^n | Ey \geq 1\}$ where E is the arc-node incidence matrix for the underlying graph of the digraph.*

Theorem 2 give rise to a characterization of the probability vectors π which can consistently be assigned to a set of logical sentences represented by a digraph. This theorem is the one which give rise to the interpretation given in this paper.

Theorem 3 *([1]) Suppose we have a set of logical formulas represented by a digraph. Then π is a consistent assignment of probabilities to the logical sentences if and only if $\pi \in conv\{y \in B^n | Ey \geq 1\}$ where E is the arc-node incidence matrix for the underlying graph of the digraph.*

Combining Theorems 1 and 3 we obtain the following result.

Theorem 4 *Suppose we have a set of logical formulas represented by a digraph. Let E denote the arc-node incidence matrix for the underlying graph of the digraph.*
Then $conv\{\pi \in B^n | E\pi \geq 1\}$ is given by the following set of linear inequalities: $(1, \pi)^t \cdot (y_0, y) \leq 0$, where (y_0, y) is an extreme ray of (5).

Actually, the result given in Theorem 4 is rather evident (and well known). It holds true for all binary matrices E. For arbitrary binary matrices E the matrix A in the system (5) should have a column for every binary solution to $E\pi \geq 1$.

4 A logical interpretation of the facets for the set packing polytope

In this section we use the results described in section 3 to obtain a logical interpretation of all the facets of the set packing polytope.

Let E be an $m \times n$ matrix with entries 0 or 1. The set packing polytope is defined as $\text{conv}\{\bar{\pi} \in B^n | E\bar{\pi} \leq 1\}$, where conv is short for convex hull.

Remark: It may seem a bit odd that we use the variables $\bar{\pi}$ in the definition of the set packing polytope instead of the usual variables x. The reason for this choice is that it will make the presentation of the results in this section more easy to understand.

Consider an equation $\bar{\pi}_1 + \bar{\pi}_2 + \ldots + \bar{\pi}_r \leq 1$. The set of binary solutions to this inequality can be expressed equivalently as the set of binary solutions to the set of inequalities $\bar{\pi}_i + \bar{\pi}_j \leq 1,\ 1 \leq i < j \leq r$. In particular it can be assumed that E has exactly two positive entries (1's) in each row (the case with less than two positive entries in a row is not interesting).

The procedure to obtain a complete logical interpretation of the facets of the set packing polytope is outlined below:

1. Formulate the set packing polytope as the convex hull $\text{conv}\{\bar{\pi} \in B^n | E\bar{\pi} \leq 1\}$ where E is an $m \times n$ $0-1$ matrix with exactly two positive entries in each row.

2. By complementing variables ($\pi_j \equiv 1 - \bar{\pi}_j, \quad j = 1, \ldots, n$) transform the problem into the problem of determining the convex hull of a set covering problem $\text{conv}\{\pi \in B^n | E\pi \geq 1\}$.

3. E is the arc-node incidence matrix for a graph $\mathcal{G}$. Give the edges in $\mathcal{G}$ arbitrary orientations. The resulting digraph represents a set of logical sentences $S_1, S_2, \ldots, S_n$. Determine the A matrix for these sentences as described in section 2, where a_{ij} is the truth value of sentence S_i in possible world j. Duplicate columns in A should be removed.

4. Using Theorem 4 a complete description of the convex hull of the set covering problem can be obtained:

   $\text{conv}\{\pi \in B^n | E\pi \geq 1\}$ is given by the following set of linear inequalities: $(1, \pi)^t \cdot (y_0, y) \leq 0$, where (y_0, y) is an extreme ray of $\{(y_0, y) | e \cdot y_0 + yA \leq 0\}$.

   We can give a logical interpretation of the facets of the set-covering problem given above:

   > Consider one of the facets, say $\pi_1 + \pi_2 + \cdots + \pi_t \geq b$, of the set covering polytope $\text{conv}\{\pi \in B^n | E\pi \geq 1\}$. This inequality states that the sum of the probabilities of the first t sentences $S_1, S_2, \ldots, S_t$ represented by the digraph should be at least b, because otherwise $\pi_1, \ldots, \pi_n$ is not a consistent assignment of probabilities to the logical sentences $S_1, S_2, \ldots, S_n$.

5. Complementing variables again a complete description of the convex hull of the set packing polytope $\text{conv}\{\bar{\pi} \in B^n | E\bar{\pi} \leq 1\}$ is obtained.

   We can give a logical interpretation of the facets of the set packing problem given above:

   > First notice that if π_j is the probability that sentence S_j is true then $\bar{\pi}_j = 1 - \pi_j$ is the probability that the complementary sentence $\bar{S}_j \equiv \neg S_j$ to sentence S_j is true. A constraint in the set packing polytope is obtained by complementing the variables in the set covering polytope. Therefore, if $\pi_1 + \pi_2 + \cdots + \pi_t \geq b$ is a constraint in the set covering polytope, then $\bar{\pi}_1 + \bar{\pi}_2 + \cdots + \bar{\pi}_t \leq t - b$ is a constraint in the set packing polytope (and vice versa). The logical interpretation of this constraint is that the sum of probabilities assigned to sentences $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_t$ should be at most $t - b$ because otherwise $\bar{\pi}_1, \ldots, \bar{\pi}_n$ it is not a consistent assignment of probabilities to the logical sentences $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n$.

The main result in this paper is Theorem 5 which states that all facets of the set packing problem can be explained logically. The proof follows directly from the above mentioned procedure.

Theorem 5 *Given a set packing polytope it is possible to give a logical interpretation of all the facets of the polytope.*

Example 3

In this example we will demonstrate how to give a logical interpretation of the facets of the set packing polytope defined by (6):

$$\begin{array}{lllllllll} \bar{\pi}_1 & + & \bar{\pi}_2 & + & \bar{\pi}_3 & & & \leq 1 \\ & & \bar{\pi}_2 & & & + & \bar{\pi}_4 & \leq 1 \\ & & & & \bar{\pi}_3 & + & \bar{\pi}_4 & \leq 1 \\ \multicolumn{8}{l}{\bar{\pi}_j \in \{0,1\},\ j = 1,\ldots,4.} \end{array} \tag{6}$$

First the set of constraints (6) is expressed equivalently as:

$$\begin{array}{llll} \bar{\pi}_1 & + & \bar{\pi}_2 & \leq 1 \\ \bar{\pi}_1 & + & \bar{\pi}_3 & \leq 1 \\ \bar{\pi}_2 & + & \bar{\pi}_3 & \leq 1 \\ \bar{\pi}_2 & + & \bar{\pi}_4 & \leq 1 \\ \bar{\pi}_3 & + & \bar{\pi}_4 & \leq 1 \\ \multicolumn{4}{l}{\bar{\pi}_j \in \{0,1\},\ j = 1,\ldots,4.} \end{array} \tag{7}$$

Complementing variables $\bar{\pi}_j = 1 - \pi_j \ j = 1,\ldots,4$ the following set of constraints results:

$$\begin{array}{llll} \pi_1 & + & \pi_2 & \geq 1 \\ \pi_1 & + & \pi_3 & \geq 1 \\ \pi_2 & + & \pi_3 & \geq 1 \\ \pi_2 & + & \pi_4 & \geq 1 \\ \pi_3 & + & \pi_4 & \geq 1 \\ \multicolumn{4}{l}{\pi_j \in \{0,1\},\ j = 1,\ldots,4.} \end{array} \tag{8}$$

The constraint matrix in (8) is the arc-node incidence matrix for the graph shown in figure 2:

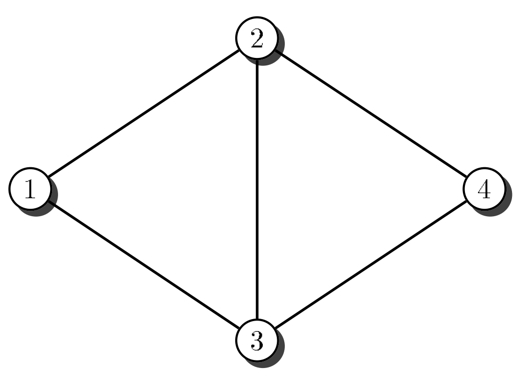

Figure 2.

Now assign arbitrary orientations to the edges in the graph in figure 2. Suppose we obtain the directed graph shown in figure 1. As shown earlier this digraph represents the set of logical sentences:

$$\begin{array}{ll} S_1: & x_1 \\ S_2: & x_1 \rightarrow x_2 \\ S_3: & (x_1 \wedge x_2) \rightarrow x_3 \\ S_4: & (x_2 \wedge x_3) \rightarrow x_4. \end{array}$$

As shown in example 2 we can determine the A matrix for this particular set of sentences, as well as the set of linear constraints the truthvalues of the sentences should fulfill. This linear system is exactly the system (8).

Using Theorem 4 we can find the convex hull of (8) by finding the extreme rays of the following set of constraints:

$$\begin{array}{ccccccccccc} y_0 & & & + & y_2 & + & y_3 & + & y_4 & \leq 0 \\ y_0 & & & + & y_2 & + & y_3 & & & \leq 0 \\ y_0 & + & y_1 & & & + & y_3 & + & y_4 & \leq 0 \\ y_0 & + & y_1 & + & y_2 & & & + & y_4 & \leq 0 \\ y_0 & + & y_1 & + & y_2 & + & y_3 & & & \leq 0 \\ y_0 & + & y_1 & + & y_2 & + & y_3 & + & y_4 & \leq 0 \\ & & & y_j \text{ free}, & j = 0, \ldots, 4 \end{array} \tag{9}$$

The extreme rays of (9) is given by:

$(2,-1,-1,-1,0), (2,0,-1,-1,-1), (-1,1,0,0,0), (-1,0,1,0,0), (-1,0,0,1,0), (-1,0,0,0,1)$.

Therefore the convex hull of (8) is determined by the set of constraints (10):

$$\begin{array}{cccccccc} - & \pi_1 & - & \pi_2 & - & \pi_3 & & & \leq -2 \\ & & - & \pi_2 & - & \pi_3 & - & \pi_4 & \leq -2 \\ & & & \pi_j \leq 1, & j = 1, \ldots, 4. \end{array} \tag{10}$$

We can give a logical explanation of the two constraints in (10):

Suppose we assign probabilities $\pi = (\pi_1, \ \pi_2, \ \pi_3, \ \pi_4)$ to the four sentences $S_1, \ \cdots, S_4$, where π_i is the probability that sentence S_i is true.

> The first constraint in (10) states that the sum of probabilities assigned to sentences S_1, S_2 and S_3 should be at least 2. If this is not so the assignment of probabilities to the sentences is not a consistent assignment.
>
> The second constraint in (10) states that the sum of probabilities assigned to sentences S_2, S_3 and S_4 should be at least 2. If this is not so the assignment of probabilities to the sentences is not a consistent assignment.

Complementing variables we obtain the convex hull to the system (7) (and (6)).

$$\begin{array}{ccccccccl} \bar{\pi}_1 & + & \bar{\pi}_2 & + & \bar{\pi}_3 & & & \leq 1 \\ & & \bar{\pi}_2 & + & \bar{\pi}_3 & + & \bar{\pi}_4 & \leq 1 \\ & & \multicolumn{5}{l}{\bar{\pi}_j \geq 0,\ j = 1, \ldots, 4.} \end{array} \tag{11}$$

We are now in a position to give a logical interpretation of the constraints in (11): Recall that if π_j is the probability that sentence S_j is true then $1 - \pi_j$ is the probability that the complementary sentence $\bar{S}_j$ to sentence S_j is true. The complementary sentences is shown below:

$$\begin{array}{ll} \bar{S}_1 : & \neg x_1 \\ \bar{S}_2 : & x_1 \wedge \neg x_2 \\ \bar{S}_3 : & x_1 \wedge x_2 \wedge \neg x_3 \\ \bar{S}_4 : & x_2 \wedge x_3 \wedge \neg x_4. \end{array}$$

> The first constraint in (11) states that the sum of probabilities assigned to sentences $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$ should be at most 1. If this is not so the assignment of probabilities to the sentences is not a consistent assignment.
>
> The second constraint in (11) states that the sum of probabilities assigned to sentences $\bar{S}_2$, $\bar{S}_3$ and $\bar{S}_4$ should be at most 1. If this is not so the assignment of probabilities to the sentences is not a consistent assignment.

□

5 Conclusions

In this paper we have shown that it is possible to give a logical explanation why all the facets of the set packing polytope are in fact necessary and sufficient in the description of the polytope. Associated with any set packing polytope is a set of logical sentences. Any of these sentences are assumed to be true with some probability. It turns out that a facet of the set packing polytope is necessary in the description of the polytope because the facet gives a necessary condition on the consistency of the probabilities assigned to the sentences.

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