A remark on the definition of B-hyperpath

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Abstract

In this note we show that a commonly used definition of a hyperpath in a directed hypergraph is not correct. This is done by presenting a counter-example which fulfils the definition but is not a hyperpath.

Keywords: Directed hypergraphs, hyperpaths, B-paths.

1 Introduction

In the last two decades, several problems arising from different application areas were modelled in terms of *hyperpaths* in *directed hypergraphs*. A general theory of directed hypergraphs was developed for the first time by Gallo *et al.* [2]. Their paper proposed a definition of hyperpath (called *B-path*) based on an intuitive concept of hyperconnection (*B-connection*). The definition aimed at characterizing the topological structure of a *minimal* sub-hypergraph B-connecting a pair of nodes. However, the characterization seems to fail in some cases. Here we present a counter-example that satisfies the definition but is *not* a B-path, i.e. it does not B-connect two nodes as supposed.
Note that the theoretical results in [2] are not affected, since they are based on the sound concept of B-connection, and do not rely on the topological characterization of B-paths. The same holds true for other papers (e.g. [3] and [4]) that adopted the definition in [2]. Several correct definitions of hyperpath have been given in the literature; however, a discussion of these definitions is not addressed here.

2 Hypergraphs, hyperconnection, hyperpaths

A *directed hypergraph* is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of nodes, and $\mathcal{E}$ is the set of (directed) *hyperarcs*. A hyperarc $e \in \mathcal{E}$ is a pair $e = (T(e), H(e))$ where $T(e) \subset \mathcal{V}$ and $H(e) \subseteq \mathcal{V} \setminus T(e)$; $T(e)$ and $H(e)$ denote the *tail* nodes and the *head* nodes, respectively. A *B-arc* is a hyperarc e such that $|H(e)| = 1$. A *B-graph* is a hypergraph of which the hyperarcs are B-arcs.
A *path* P_{st} in a hypergraph $\mathcal{H}$ is a sequence of nodes and hyperarcs in $\mathcal{H}$:

$$P_{st} = (v_1 = s, e_1, v_2, e_2, ..., e_q, v_{q+1} = t)$$

where $v_1 \in T(e_1)$, $v_{q+1} \in H(e_q)$ and $v_i \in H(e_{i-1}) \cap T(e_i)$ for $i = 2, ...q$. A node v is *connected* to node u if a path P_{uv} exists in $\mathcal{H}$. A *cycle* is a path P_{st} where $t \in T(e_1)$. A path is *cycle-free*

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if it does not contain any subpath which is a cycle, i.e. $v_i \in T(e_j) \Rightarrow j \geq i$, $1 \leq i \leq q+1$. If $\mathcal{H}$ contains no cycles, it is *acyclic*.
The concept of *B-connection* in hypergraphs is captured by the following intuitive definition; compare Proposition 3.1 in [2].

Definition 1 B-connection to node s in a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$

1. Node s is B-connected to itself;
2. If for some $e \in \mathcal{E}$ all the nodes in $T(e)$ are B-connected to s then each node $u \in H(e)$ is B-connected to s.

The concept of *B-hyperpath*, or simply *B-path*, generalizes the notion of simple path in a directed graph. A *B-path* from node s to node t in a hypergraph $\mathcal{H}$ is a *minimal* sub-hypergraph of $\mathcal{H}$ where t is B-connected to s according to Definition 1. Here, minimality is intended with respect to the deletion of nodes and hyperarcs.
A B-path can be defined as a *sequence* of hyperarcs used to prove that t is B-connected to s: see e.g. [1]. The following topological characterization of B-paths, not directly related to Definition 1, has been proposed in [2].

Definition 2 A *B-path* π_{st} from s to t in $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a minimal sub-hypergraph $\mathcal{H}_\pi = (\mathcal{V}_\pi, \mathcal{E}_\pi)$ satisfying the following conditions:

1. $\mathcal{E}_\pi \subseteq \mathcal{E}$
2. $s, t \in \mathcal{V}_\pi = \bigcup_{e \in \mathcal{E}_\pi} (T(e) \cup H(e))$
3. $u \in \mathcal{V}_\pi \setminus \{s\} \;\Rightarrow\; u$ is connected to s in $\mathcal{H}_\pi$ by means of a cycle-free simple path.

Unfortunately, Definition 2 is too weak, also if B-graphs are considered; a counterexample is provided by B-graph $\mathcal{H}_{st}$ in Figure 1. It can be shown that $\mathcal{H}_{st}$ fulfils Definition 2; for example, it contains a cycle-free path from node s to node v_4, namely $(s, e_1, v_1, e_2, v_3, e_4, v_4)$. However, node t is not B-connected to s in $\mathcal{H}_{st}$, according to Definition 1; the reader can easily check that only node v_1 is B-connected to s.

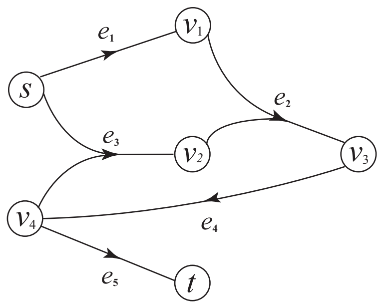

Figure 1: A counterexample: B-graph $\mathcal{H}_{st}$

Note that $\mathcal{H}_{st}$ contains a cycle. As long as B-graphs are considered, Definition 2 can be made correct by further imposing that π_{st} must be acyclic; equivalently, Definition 2 is correct for acyclic B-graphs: see Property 2.1 in [4]. Remark that a B-path in a general hypergraph is not required to be acyclic, as shown in [2], Figure 5(a).

3 Conclusion

We have shown that a topological characterization of B-paths proposed in the literature is not correct, unless acyclicity is imposed. It remains an open question to find a concise and elegant characterization of B-paths, valid for B-graphs as well as general hypergraphs.

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