

A hierarchical Markov decision process modeling feeding and marketing decisions of growing pigs

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Abstract: Feeding is the most important cost in the production of growing pigs and has a direct impact on the marketing decisions, growth and the final quality of the meat. In this paper, we address the sequential decision problem of when to change the feed-mix within a finisher pig pen and when to pick pigs for marketing. We formulate a hierarchical Markov decision process with three levels representing the decision process. The model considers decisions related to feeding and marketing and finds the optimal decision given the current state of the pen. The state of the system is based on information from on-line repeated measurements of pig weights and feeding and is updated using a Bayesian approach. Numerical examples are given to illustrate the features of the proposed optimization model.

Keywords: OR in agriculture; stochastic programming; hierarchical Markov decision process; herd management; Bayesian updating.

1 Introduction

In production systems of growing pigs, feeding is the most important operation and has a direct influence on the cost and the quality of the meat. Another important operation is the timing of *marketing*. It refers to a sequence of culling decisions until the production unit is empty. As a result the profit of the production unit is highly dependent on the feeding cost and on good timing of marketing, i.e. decisions about feeding and marketing have a direct impact on profit.

In a production system of growing/finishing pigs (Danish standards), the animals may be considered at different levels: herd, section, pen, or animal. The herd is a group of sections, a section includes some pens, and a finisher pen involves some animals (usually 15-20). New piglets are transferred to a weaner unit approx. four weeks after birth, and they stay for approx.

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eight weeks until they weigh approx. 30 kg. The pigs are then moved to a finisher pen where they grow until marketing (9-12 weeks). In the finisher pen, the farmer should determine which pigs should be selected for slaughter (individual marketing). The reward of marketing a pig depends on the unit meat price of the carcass weight and the leanness of the pig. The meat price is highest if the carcass weight of the pig lies in a specific interval. Next, after a sequence of individual marketings, the farmer must decide when to *terminate* (empty) the rest of the pen. Terminating a pen means that the remaining pigs in the pen are sent to the slaughterhouse (in one delivery) and after cleaning the pen, another group of piglets (each weighing approx. 30 kg) is inserted into the pen and *the production cycle* is repeated. That is, the farmer must time the marketing decisions while simultaneously considering the carcass weight in relation to the best interval, the leanness, and the length of the production cycle. For an extended overview over pig production of growing pigs, see Pourmoayed and Nielsen (2014a).

The growth and leanness of the pigs will be highly dependent on the feed given. Phase feeding is a common method used in the production of the growing pigs. In the finisher pen the growing period typically includes 3 or 4 phases and each phase involves a predefined *feed-mix* which is a mixture of different ingredients (barley, soy, maize, etc.). A relevant decision is when to change the current feed-mix (transition to a new phase) and what type of feed-mix to use in the next phase.

Since the choice of feed-mix affects the pigs' growth, a specific feeding strategy has an impact on the marketing strategy. That is, the economic optimization of feeding and marketing decisions is interrelated and requires a simultaneous analysis. Consequently, a sequential decision model is needed that considers both feeding and marketing decisions. To the best of our knowledge, there are only a few studies that take into account these decisions simultaneously (Niemi, 2006; Sirisatien, Wood, Dong, and Morel, 2009). However, these studies consider the problem at animal level and do not take into account the inhomogeneity of animals in growth and feed intake. The aim of this paper is to close this gap and consider the problem at pen level instead.

In this paper we formulate a hierarchical Markov decision process which takes into account decisions related to feeding and marketing of growing pigs at pen level. We assume that the production is cyclic, i.e. when the pen is emptied, not only a regular state transition takes place, but rather the process (the current batch of pigs) is restarted.

The model considers time series of pig weights and feeding obtained from online monitoring, e.g. from a set of sensors in the pen. A Bayesian approach is used to update the state of the system such that it contains the relevant information based on the previous measurements. More precisely, two state space models for *Bayesian forecasting* (West and Harrison, 1997) are used to update the estimates of live weights and feed intake on a weekly basis.

The structure of the paper is as follows. First, Section 2 gives a short literature review. Second, a detailed description of the optimization model is given in Section 3. Next, Section 4 presents the statistical models which are embedded into the model. In Section 5, numerical examples are considered to show the functionality of the proposed optimization model. Finally, conclusions and directions for further research are given in Section 6.

2 Literature review

Due to the dynamic nature of the production environment of growing pigs, the marketing and feeding decisions are sequential, complex and hard to optimize. Various models have been considered to deal with this complexity.

Some studies consider only the marketing decisions. Chavas, Kliebenstein, and Crenshaw (1985) applied the concepts of optimal control theory to find the optimal time of marketing of individual animals. Jørgensen (1993) used a dynamic programming approach to optimize a given heuristic framework for delivering the pigs to the slaughterhouse. Boland, Preckel, and Schinckel (1993) considered the optimal slaughter pig marketing problem under different pricing models and for each pricing system, they found the optimal slaughter weight. Kure (1997) considered the problem at batch level and used the replacement theory concepts and a recursive dynamic programming method to determine the optimal time of marketing the pigs. Toft, Kristensen, and Jørgensen (2005) optimized both marketing and treatment decisions (e.g. regarding vaccination for disease problems) using a *hierarchical Markov decision process (HMDP)*. Boys, Li, Preckel, Schinckel, and Foster (2007) implemented a simulation approach to determine the best marketing strategy to utilize full truck capacity for delivering the pigs to the packers. In the study by Ohlmann and Jones (2008), a mixed integer programming model was proposed to find the best marketing strategy under an annual profit criterion. Kristensen, Nielsen, and Nielsen (2012) suggested a two-level HMDP to find the best marketing strategy according to the data from an online monitoring system.

Other studies focus on sequential feeding decisions, i.e. finding the best strategy for choosing the appropriate feed-mix during the growing period of animals. One example is Glen (1983) who proposed a dynamic programming approach to determine the sequence of feed-mixes in the production unit. In the study by Boland, Foster, and Preckel (1999), a linear programming approach was used to specify the optimal time of changing the feed-mix and also the optimal nutrient ingredients of the feed-mix. A genetic algorithm was applied by Alexander, Morel, and Wood (2006) to find the best nutrient components of each feed-mix.

Only few studies take both marketing and feeding decisions into account. Niemi (2006) used a mechanistic function to model the animal growth trend during the growing period. Niemi (2006) further applied a stochastic dynamic programming method to find the best nutrient ingredients and also the best time of marketing. In the study by Sirisatien et al. (2009), a genetic algorithm was used. Each iteration resulted in a set of feeding schedules followed by the optimal values of the nutrient ingredients and feeding period. Both studies considered the problem at animal level and did not take into account the inhomogeneity of animals with respect to growth and feed conversion rate.

Markov decision models are a well-known modeling technique within animal science used to model livestock systems. See for instance Rodriguez, Jensen, Pla, and Kristensen (2011) and Nielsen, Jørgensen, Kristensen, and Østergaard (2010). For a recent survey see Nielsen and Kristensen (2015), which cites more than 100 papers using (hierarchical) Markov decision processes to model and optimize livestock systems. An HMDP is an extension of a semi Markov decision process (semi-MDP) where a series of finite-horizon semi-MDPs are combined into one process at the founder level called the main process (Kristensen, 1988; Kristensen and Jørgensen, 2000).

As a result the state space at the founder level can be reduced and larger models can be solved using a modified policy iteration algorithm under different criteria (Nielsen and Kristensen, 2015). Modeling the problem using an HMDP compared to a semi-MDP contributes to reducing the *curse of dimensionality*, since the total number of state variables can be decreased. Moreover, the total number of states at the founder level is lower (i.e. the matrix which must be inverted in the modified policy iteration algorithm is much smaller).

A *state space model (SSM)* (West and Harrison, 1997) is a statistical model which may be used to transform large datasets obtained using online sensors into the required information about the production process. An SSM consists of a set of latent variables and a set of observed variables. At a specified point in time the conditional distribution of the observed variables is a function of the latent variables specified via the observation equations. The latent variables change over time as described via the system equations. The observations are conditionally independent given the latent variables. Thus the estimated value of the latent variables at a time point may be considered as the state of the system, and with Bayesian forecasting (the Kalman filter) we can estimate the latent variables/real state of the system via the observed variables. Examples of SSMs applied to agricultural problems are Bono, Cornou, and Kristensen (2012); Cornou, Vinther, and Kristensen (2008) and Bono, Cornou, Lundbye-Christensen, and Kristensen (2013). Moreover, an SSM can be discretized and embedded into an HMDP (Nielsen, Jørgensen, and Højsgaard, 2011).

3 Model description

Our pig marketing and feeding problem is modeled using a *hierarchical Markov decision process (HMDP)* with three levels. A short introduction to HMDPs is given below. As techniques from both statistical forecasting and operations research are used, consistent notation can be hard to specify. To assist the reader, Appendix A provides an overview.

An HMDP is an extension of a semi-MDP where a series of finite-horizon semi-MDPs are combined into one infinite time-horizon process at the founder level called the *founder process* (Kristensen and Jørgensen, 2000). The idea is to expand the stages of a process to so-called *child processes*, which again may expand stages further to new child processes leading to multiple levels. At the lowest level the HMDP consists of a set of finite-horizon semi-MDPs (see e.g. Tijms, 2003, Chap. 7). All processes are linked together using jump actions (see Figure 1).

A *finite-horizon semi-MDP* considers a sequential decision problem over \mathcal{N} stages. Let \mathbb{I}_n denote the finite set of system states at stage n . When *state* $i \in \mathbb{I}_n$ is observed, an *action* a from the finite set of allowable actions $\mathbb{A}_n(i)$ must be chosen, and this decision generates *reward* $r_n(i, a)$. Moreover, let $u_n(i, a)$ denote the *stage length* of action a , i.e. the expected time until the next decision epoch (stage $n + 1$) given action a and state i . Finally, let $\Pr(j | n, i, a)$ denote the *transition probability* of obtaining state $j \in \mathbb{I}_{n+1}$ at stage $n + 1$ given that action a is chosen in state i at stage n .

An HMDP with three levels is illustrated in Figure 1 using a *state-expanded hypergraph* (Nielsen and Kristensen, 2006). At the first level, a single *founder process* \mathfrak{p}^0 is defined. Index 0 indicates that the process has no ancestral processes. We assume that \mathfrak{p}^0 is running over an



Figure 1: An illustration of a stage in an HMDP. At the founder level (Level 0) we have a single infinite-horizon founder process p^0 . A child process, such as p^1 at Level 1 (oval box), is uniquely defined by a given stage, state (node), and action (hyperarc) of its parent process and linked with the parent process using its initial probability distribution (solid lines) and its terminating actions (dashed lines). Each process at level 2 is a semi-MDP. Note that only a subset of the actions is drawn.

infinite number of stages and that all stages have identical state and action spaces and hence just a single stage is illustrated in Figure 1. Let p^{l+1} denote a *child process* at level $l+1$. Process p^{l+1} is uniquely defined by a given stage n^l , state i^l and action a^l of *parent process* p^l . For instance, the semi-MDP p^2 in Figure 1 is defined at stage $n^1 = 2$, state i^1 and action a^1 of the process p^1 symbolized by the notation $p^2 = (p^1 \parallel (n^1, i^1, a^1))$. Each process is connected to its parent and child processes using *jump actions* which can be divided into two groups, namely, a *child jump action* that represents an *initial probability distribution* of transitions to a child process or a *parent jump action* that represents a *terminating probability distribution* of transitions to a parent process. This is illustrated in Figure 1 for process p^1 where child jump action a^1 (illustrated using a solid hyperarc) represents a transition to the child process p^2 and parent jump action a^2 (illustrated using a dashed hyperarc) represents termination of the process p^2 . Like traditional actions, jump actions are associated with an expected reward, action length, and a set of transition probabilities. Each node in Figure 1 at a given stage n of a process p^l corresponds to a state in \mathbb{I}_n^l . For example, there are 3 states at stage 1 in process p^2 . Similarly each *hyperarc* corresponds to an action, e.g. action a (gray hyperarc) results in a transition to either state j_1 or j_2 .

A policy is a decision rule/function that assigns to each state in a process a (jump) action. That is, choosing a policy corresponds to choosing a single hyperarc out of each node in Figure 1. Given a policy, the reward at a stage of a parent process equals the total expected rewards of the corresponding child processes. For instance, in Figure 1, the reward of choosing action a^1 in state i^1 at stage $n^1 = 2$ in process p^1 equals the total expected reward of process p^2 . A similar

approach can be used to calculate the transition probabilities and the stage length of an action at a stage of a parent process.

Different optimality criteria may be considered. In this paper, our optimality criterion is to maximize the *expected reward per time unit* and the optimal policy of the HMDP can be found using a modified policy iteration algorithm. For a detailed description of the algorithm, the interested reader may consult Nielsen and Kristensen (2015).

3.1 Assumptions

Consider the problem of optimizing feeding and marketing decisions in a finisher pig pen. The problem can be modeled as a three-level semi-HMDP under the following assumptions:

- q^{\max} pigs are inserted into the finisher pen;
- a finite set of feed-mixes \mathbb{F} is available and feed-mix $f \in \mathbb{F}$ cannot be changed before it has been used for at least t_f^{\min} weeks (for simplicity, t_f^{\min} is the same for all feed-mixes);
- at most b^{\max} feeding phases can be used;
- marketing of pigs is started in week t^{\min} at the earliest;
- the pen is terminated in week t^{\max} at the latest, i.e. the maximum life time of a pig in the pen is t^{\max} ;
- the growth of a pig is independent of the other pigs in the pen, i.e. the growth is not dependent on the number of pigs in the pen;
- weekly deliveries to the abattoir in the marketing period are based on a cooperative agreement where culled pigs from each pen are grouped in one transportation delivery at a fixed time each week, i.e. the transportation cost is fixed.

To give a complete description of the three-level HMDP with feeding and marketing decisions, each semi-MDP must be specified at all levels, i.e. stages, states, and (jump) actions including the corresponding rewards, stage lengths (measured in weeks), and transition probabilities.

3.2 Stages, states and actions

As illustrated in Figure 1, the founder process of the HMDP is an infinite time-horizon process where a stage represents a life of q^{\max} pigs inserted into the pen (until termination). A stage of the process at the second level corresponds to a feeding phase in which the pigs are fed a specific feed-mix f . Finally, a stage at the third level is a week of the current production cycle in the pen under the specific feed-mix. The length, stage, states, and (jump) actions of each process at the different levels are described below. Whenever, the level is clear from the context, the superscript indicating the current level under consideration will be left out to avoid heavy notation.

Level 0 - Founder process p^0

Stage: A production cycle of q^{\max} pigs, i.e. from inserting the piglets into the pen until terminating the pen.

Time horizon: Infinite (since the number of filling and emptying a pen is infinite).

States: A single state representing the start of a production cycle ($\mathbb{I} = \{i^0\}$).

Actions: One child jump action a^0 representing insertion of a new group of piglets ($\mathbb{A}(i^0) = \{a^0\}$).

Level 1 - Parent process $p^1 = (p^0 \parallel (n^0, i^0, a^0))$

Stage: A feeding phase with a given feed-mix.

Time horizon: Given a maximum of b^{\max} feeding phases, the maximum number of stages in process p^1 is $\mathcal{N} = b^{\max} + 1$ since a dummy stage is added at the end.

States: First, consider stage/feeding phase $2 \leq n \leq b^{\max}$. A state i is defined using the following state variables:

f_n : previous feed-mix (feed-mix in stage/phase $n - 1$);

t_n : starting time of phase (week);

q_n : number of pigs in the pen at the beginning of stage/phase n ;

w_n : model information related to the weight of the pigs, obtained using Bayesian updating ($w_n \in \mathbb{W}_n$). Section 4 provides details on the way the information is obtained.

Furthermore, at this level, a dummy state \tilde{i} is added to represent pen termination. Note that due to the model assumptions, the earliest starting time of phase n is $(n - 1)t_f^{\min} + 1$. Moreover, if $t_n \leq t^{\min}$ then $q_n = q^{\max}$. Hence the set of states becomes

$$\mathbb{I}_n = \{i = (f_n, t_n, q_n, w_n) \mid f_n \in \mathbb{F}, t_n \in \{(n - 1)t_f^{\min} + 1, \dots, t^{\max} - 1\}, \\ q_n \in \{q^{\max} \mathbf{I}_{\{t_n \leq t^{\min}\}} + \mathbf{I}_{\{t_n > t^{\min}\}}, \dots, q^{\max}\}, w_n \in \mathbb{W}_n\} \cup \{\tilde{i}\},$$

where $\mathbf{I}_{\{\cdot\}}$ denotes the indicator function.

Next, consider stage $n = 1$. Here the number of states to $\mathbb{I}_n = \mathbb{W}_n$ can be reduced, since $t_n = 1$, $q_n = q^{\max}$, and there is no previous feed-mix.

Finally, at the dummy stage ($n = \mathcal{N}^1$), only the dummy state \tilde{i} representing pen termination is defined.

Actions: At stage $n = 1$, it is possible to choose a feed-mix $f \in \mathbb{F}$ at state $i = w_n$, i.e. the set of child jump actions is $\mathbb{A}_n(i) = \{a_f \mid f \in \mathbb{F}\}$. At the subsequent stages ($1 < n < \mathcal{N}^1$), possible child jump actions at state $i = (f_n, t_n, q_n, w_n)$ are $\mathbb{A}_n(i) = \{a_f \mid f \in \mathbb{F} \setminus \{f_n\}\}$. The

length of all child jump actions choosing a feed-mix is zero. In the dummy state \tilde{i} a single dummy parent jump action \tilde{a} with length zero is considered which represents that the pen has been terminated.

Level 2 - MDP $p^2 = (p^1 \parallel (n^1, i^1, a^1))$

At the lowest level a semi-MDP is defined for each stage/feeding phase n^1 , parent state $i^1 = (f_{n^1}, t_{n^1}, q_{n^1}, \mathbb{w}_{n^1})$, and action $a^1 = a_f$ corresponding to choosing feed-mix f .

Stage: A week in the current feeding phase.

Time horizon: A stage is defined for each week t_{n^1}, \dots, t^{\max} and hence the time horizon becomes $\mathcal{N} = t^{\max} - t_{n^1} + 1$. That is, stage $n = 1, \dots, \mathcal{N}$ corresponds to week $t_{n^1} + n - 1$ ($n - 1$ weeks since the feed-mix was changed).

States: Given stage n , a state i consists of the following state variables:

q_n : number of pigs in the pen at the beginning of the week;

\mathbb{w}_n : model information related to the weight of the pigs, obtained using Bayesian updating ($\mathbb{w}_n \in \mathbb{W}_n$);

\mathbb{g}_n : model information related to the growth of the pigs, obtained using Bayesian updating ($\mathbb{g}_n \in \mathbb{G}_n$). Further details on how \mathbb{g}_n and \mathbb{w}_n are obtained, are given in Section 4.

A dummy state \tilde{i} is also added to represent pen termination. Therefore the set of states becomes:

$$\mathbb{I}_n = \{i = (q_n, \mathbb{w}_n, \mathbb{g}_n) \mid q_n \in \{q^{\max} \mathbf{I}_{\{t_{n^1} + n - 1 \leq t^{\min}\}} + \mathbf{I}_{\{t_{n^1} + n - 1 > t^{\min}\}}, \dots, q^{\max}\}, \\ \mathbb{w}_n \in \mathbb{W}_n, \mathbb{g}_n \in \mathbb{G}_n\} \cup \{\tilde{i}\}.$$

Actions: Consider state $i = (q_n, \mathbb{w}_n, \mathbb{g}_n)$ at stage n . If marketing is not possible at this stage (since $t_{n^1} + n - 1 < t^{\min}$), then the production process continues for another week with the current feed-mix using action a_{cont} . If marketing is possible ($t_{n^1} + n - 1 \geq t^{\min}, n < \mathcal{N}^1$), then the set of actions can be expanded to the parent jump action a_{term} where the pen is terminated and actions a_q , which implies that the q heaviest pigs are culled (individual marketing). If $n > t_f^{\min}$ the current feed-mix can be changed, which corresponds to parent jump action a_{newMix} . Finally, at the last stage $n = \mathcal{N}$, the pen must be terminated. Hence the set of actions becomes

$$\mathbb{A}_n(i) = \begin{cases} \{a_{\text{cont}}\}, & t_{n^1} + n - 1 < t^{\min}, n \leq t_f^{\min}, \\ \{a_{\text{cont}}, a_{\text{newMix}}\}, & t_{n^1} + n - 1 < t^{\min}, t_f^{\min} < n < \mathcal{N}, \\ \{a_{\text{cont}}, a_{\text{term}}\} \cup \{a_q \mid 1 \leq q < q_n\}, & t_{n^1} + n - 1 \geq t^{\min}, n \leq t_f^{\min}, \\ \{a_{\text{cont}}, a_{\text{newMix}}, a_{\text{term}}\} \cup \{a_q \mid 1 \leq q < q_n\}, & t_{n^1} + n - 1 \geq t^{\min}, t_f^{\min} < n < \mathcal{N}, \\ \{a_{\text{term}}\}, & n = \mathcal{N}. \end{cases}$$

The lengths of actions a_{cont} and a_q are one week while the lengths of actions a_{term} and a_{newMix} are zero. State \tilde{i} has a single dummy parent jump action \tilde{a} of length zero.

3.3 Transition probabilities

To complete the formulation of the HMDP, transition probabilities must be specified for all (jump) actions.

Level 0 - Founder process p^0

Given state i^0 and child jump action a^0 (insertion of a new group of piglets), a transition to state $i^1 = \mathbb{w}_1$ at the first stage ($n^1 = 1$) of process p^1 happens with probability $\Pr(i^1 | i^0, a^0) = \Pr_0(\mathbb{w}_1)$, where $\Pr_0(\mathbb{w}_1)$ denotes the initial probability of weight information \mathbb{w}_1 .

Level 1 - Parent process p^1

Consider state $i = (f_n, t_n, q_n, \mathbb{w}_n)$ and child jump action $a = a_f$ that corresponds to choosing a specific feed-mix $f \in \mathbb{F}$. A transition to state $i^2 = (\tilde{q}_1, \tilde{\mathbb{w}}_1, \tilde{\mathbb{g}}_1)$ at the first stage ($n^2 = 1$) of process p^2 happens with probability

$$\Pr(i^2 | n, i, a) = \begin{cases} \Pr_0(\tilde{\mathbb{g}}_1 | f), & \tilde{q}_1 = q_n, \tilde{\mathbb{w}}_1 = \mathbb{w}_n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Pr_0(\tilde{\mathbb{g}}_1 | f)$ denotes the initial probability of growth information for state $\tilde{\mathbb{g}}_1$ given feed-mix f . For dummy state \tilde{i} and parent jump action \tilde{a} , a deterministic transition to state i^0 happens.

Level 2 - Semi-MDP $p^2 = (p^1 \parallel (n^1, i^1, a^1))$

First, consider state $i = (q_n, \mathbb{w}_n, \mathbb{g}_n)$ in process p^2 starting at week t_{n^1} , given $a^1 = a_f$, i.e. the process uses feed-mix f . At Level 2, two parent jump actions are considered. If the feed-mix is changed ($a = a_{\text{newMix}}$), then the process terminates and makes a deterministic transition to state $i^1 = (f, t_{n^1} + n - 1, q_n, \mathbb{w}_n)$ at stage $n^1 + 1$. If the process is terminated using parent jump action a_{term} , then the system makes a deterministic transition to state $i^1 = \tilde{i}$ in level 1.

Next, consider states $i = (q_n, \mathbb{w}_n, \mathbb{g}_n)$ at stage n and $j = (q_{n+1}, \mathbb{w}_{n+1}, \mathbb{g}_{n+1})$ at stage $n + 1$. Two types of actions are possible. If the current feed-mix is not changed, the transition probability equals

$$\Pr(j | i, a_{\text{cont}}) = \begin{cases} \Pr(\mathbb{w}_{n+1}, \mathbb{g}_{n+1} | \mathbb{w}_n, \mathbb{g}_n), & q_{n+1} = q_n, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

and if q pigs are culled, the transition probability equals:

$$\Pr(j | i, a_q) = \begin{cases} \Pr(\mathbb{w}_{n+1}, \mathbb{g}_{n+1} | \mathbb{w}_n, \mathbb{g}_n), & q_{n+1} = q_n - q, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

The probability $\Pr(\mathbb{w}_{n+1}, \mathbb{g}_{n+1} \mid \mathbb{w}_n, \mathbb{g}_n)$ depends on the statistical models used for Bayesian forecasting and will be given in Section 4.

Finally, if the dummy parent action in state \tilde{i} is considered, a deterministic transition to state $i^1 = \tilde{i}$ in process \mathfrak{p}^1 occurs.

3.4 Expected rewards

To finalize the description of the model, the expected reward of each (jump) action must be specified.

Level 0 - Founder process \mathfrak{p}^0

Action a^0 represents the insertion of q^{\max} piglets and hence the reward equals $r(i^0, a^0) = -c_{\text{pig}}q^{\max}$, where c_{pig} denotes the unit cost of a piglet.

Level 1 - Parent process \mathfrak{p}^1

The reward of child jump action a_f (choose feed-mix f) is zero since the cost of reconfiguring the feeding system is added in Level 2. The same holds for the parent jump action \tilde{a} where the reward is assumed to be zero.

Level 2 - MDP $\mathfrak{p}^2 = (\mathfrak{p}^1 \parallel (n^1, i^1, a^1))$

The reward of choosing a new feed-mix (parent jump action a_{newMix}) is $-c_{\text{newMix}}$ where c_{newMix} denotes the fixed cost of changing from one feed-mix to another. The reward of the dummy parent jump action \tilde{a} is zero.

For the remaining actions $(a_{\text{cont}}, a_{\text{term}}, a_q)$ the expected reward equals the expected revenue from selling the pigs minus the expected cost of feeding the pigs conditioned on the values of the state variables and the action. Let $(w_{(1)}, z_{(1)}), \dots, (w_{(q_n)}, z_{(q_n)})$ denote the weight and weekly feed intake of the pigs ordered such that $w_{(k)} \leq w_{(k+1)}$. That is, $w_{(k)}$ is the weight of k th pig, i.e. the k th order statistics. If the q heaviest pigs are culled, the revenue becomes

$$\sum_{j=q_n-q+1}^{q_n} \tilde{w}_{(j)} \cdot p(\tilde{w}_{(j)}, \check{w}_{(j)}), \quad (3)$$

where $\tilde{w}_{(j)}$ and $\check{w}_{(j)}$ denote the carcass weight (kg) and the leanness (non-fat percentage) of the j th pig in the pen, respectively. Price function $p(\cdot)$ is the unit price of the meat. Similarly, the cost of feeding the $q_n - q$ lightest pigs is

$$\sum_{j=1}^{q_n-q} z_{(j)} \cdot c_f, \quad (4)$$

where c_f denotes the unit cost of feed-mix f . The expected reward $r_n(i, a_q)$ can now be found as the difference between the expected value of Equations (3) and (4). Actions a_{cont} and a_{term}

may be considered as extreme culling decisions ($q = 0$ and $q = q_n$), i.e. $r_n(i, a_{\text{cont}})$ equals the expectation of (4) with $q = 0$ and $r_n(i, a_{\text{term}})$ equals the expectation of (3) with $q = q_n$.

To evaluate the expected reward of (3) and (4), statistical models are needed to transform the repeated measurements of weight and feed intake into relevant information about weight and growth using Bayesian forecasting. This will be the focus in the next section.

4 Bayesian updating of weight and growth

In animal production, online monitoring is a relevant method to obtain data for tracking the changes and can be done regularly by sensors placed in the production units. Two types of online sensors are considered in the finisher pen which provide data about live weight and feed intake, respectively. To transform these data into information about weight and growth, we need a statistical model. In this paper state space models (SSMs) are used to estimate the mean weight μ_t and growth g_t of the pigs in the pen at time t . The same holds for the standard deviation σ_t of the pig weights in the pen.

The set \mathbb{W}_n in the HMDP will therefore contain discretized estimates of the mean weight and standard deviation and the set \mathbb{G}_n will include discretized estimates of the mean growth.

SSMs can be categorized into different groups based on the dynamic nature of the considered system and the probability distribution assumed. Two kinds of SSMs are considered and later embedded into the HMDP. In the first model, the probability distribution of the observations, related to the online sensors, is Gaussian (*GSSM*) and in the second model, these observations come from a non-Gaussian distribution (*nGSSM*). The dynamics of the system is modeled by linear equations in both models.

The next sections, first provide a description of the two models and afterwards use the models to calculate the reward and transition probabilities of the HMDP. For a short introduction to SSMs and the theorems used for Bayesian updating see Appendix B.

4.1 A GSSM for average weight and growth estimations

Let $(\hat{w}_1, \dots, \hat{w}_d)_t$ denote d weight estimates obtained by an online weighting method (e.g. image processing) at time t . That is, an estimate of the average weight at time t is $\bar{w}_t = \sum_{k=1}^d \hat{w}_k / d$. Moreover, assume that the average feed intake per pig \bar{z}_t , given feed-mix f , is measured using an automatic feeding system. The following GSSM is used to model mean weight and growth:

$$\text{system equation } (\theta_t = G\theta_{t-1} + \omega_t) : \quad \begin{pmatrix} \mu_t \\ g_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ g_{t-1} \end{pmatrix} + \begin{pmatrix} \omega_{1t} \\ \omega_{2t} \end{pmatrix}, \quad (5)$$

$$\text{observation equation } (y_t = F_t' \theta_t + v_t) : \quad \begin{pmatrix} \bar{w}_t \\ \bar{z}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k_{1t} & k_{2t} \end{pmatrix} \begin{pmatrix} \mu_t \\ g_t \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}, \quad (6)$$

The system equation (5) models the relation between the latent variables $\theta_t = (\mu_t, g_t)'$ where the first equation in (5) states that the mean weight μ_t in the pen at time t equals the mean weight at time $t - 1$ plus the mean growth and some noise. The second equation states that the mean

growth g_t in the pen follows a random process. The system noise is $\omega_t \sim N(0, W)$ and the prior distribution is $\theta_0 \sim N(m_{0,f}, C_{0,f})$ given a fixed feed-mix f .

The observation equation (6) illustrates the relation between the observed variables $y_t = (\bar{w}_t, \bar{z}_t)'$ and the latent variables. That is, in the first equation, the observed average weight equals the mean weight plus the measurement error of the weighing method, and in the second equation, the observed average feed intake equals

$$\bar{z}_t = k_{1t}\mu_t + k_{2t}g_t + v_{2t},$$

where k_{1t} and k_{2t} are two known parameters. This relation is based on a linear approximation of the relation between feed intake and growth stated in Jørgensen (1993) where k_{1t} is a dynamic parameter to cover the non-linearity of the weight term. The observation error is assumed to be $v_t \sim N(0, V)$.

Let $\mathbb{D}_t = (y_1, \dots, y_t, m_{0,f}, C_{0,f})$ denote the information available up to time t . When new values of the observable variable $y_t = (\bar{w}_t, \bar{z}_t)'$ are received, Bayesian updating (Theorem 1 in Appendix B) can be used to update the posterior $(\theta_t | \mathbb{D}_t) \sim N(m_t, C_t)$ at time t . That is, the posterior mean and covariance given the current feed-mix f become

$$m_t = \begin{pmatrix} \hat{\mu}_t \\ \hat{g}_t \end{pmatrix}, \quad C_t = \begin{pmatrix} C_{t,1} & C_{t,12} \\ C_{t,12} & C_{t,2} \end{pmatrix}.$$

The estimated means $(\hat{\mu}_t, \hat{g}_t)'$ are the best estimate of the latent variables $(\mu_t, g_t)'$. The starting time of the GSSM is when the pigs are inserted into the pen, i.e. the prior mean of the latent variable is $m_{0,f} = (\hat{\mu}_0, \hat{g}_{0,f})$ where $\hat{\mu}_0$ denotes the average weight of the piglets at insertion and $\hat{g}_{0,f}$ is the estimated growth rate given feed-mix f (prior to receiving sensor data). The initial covariance $C_{0,f}$ contains the initial covariance components of live weight and growth rate at the time of insertion given feed-mix f .

If the feed-mix is changed at time t to a new feed-mix f , this change is interpreted as a system intervention (Kristensen, Jørgensen, and Toft, 2010, Section 8.2.5) and the posterior mean and covariance are modified to

$$m_t = \begin{pmatrix} \hat{\mu}_t \\ \hat{g}_{0,f} \end{pmatrix}, \quad C_t = \begin{pmatrix} C_{t,1} & C_{0,f,12} \\ C_{0,f,12} & C_{0,f,2} \end{pmatrix},$$

where $\hat{g}_{0,f}$ denotes the initial estimate of the growth rate of the new feed-mix (prior to receiving sensor data for feed-mix f) and $C_{0,f}$ denotes the initial covariances for the feed-mix f .

4.2 An nGSSM to estimate the variance of weights in the pen

Assuming d weight estimates $(\hat{w}_1, \dots, \hat{w}_d)_t$ at time t , the unbiased sample variance at time t is $s_t^2 = \sum_{k=1}^d (\hat{w}_k - \bar{w}_t)^2 / (d-1)$. It is well known that if s_t^2 is based on observations from a normal distribution with true variance $(\sigma_t)^2$, then $(d-1)s_t^2 / (\sigma_t)^2$ follows a chi-square distribution with $d-1$ degrees of freedom (Wackerly, Mendenhall, and Scheaffer, 2008, p357). Hence the sample

variance s_t^2 follows a gamma distribution with shape α_t and scale \mathfrak{b}_t given as

$$\alpha_t = \frac{d-1}{2}, \quad \mathfrak{b}_t = \frac{2(\sigma_t)^2}{d-1}.$$

Note that since d is constant, α_t is constant and known for all $t > 1$.

An nGSSM can now be defined with observation $y_t = s_t^2$ and latent variable $\theta_t = (\sigma_t)^2$ at time t where y_t follows a gamma distribution with shape α_t and scale \mathfrak{b}_t . The natural parameter becomes $\eta_t = -1/(\sigma_t)^2$ and the impact on the latent variable ($g(\eta_t) = F_t' \theta_t$) is defined as $g(\eta_t) = (\sigma_t)^2$, i.e. $g(\eta_t) = -1/\eta_t, F_t' = 1$ (see Appendix B). The system equation is:

$$(\sigma_t)^2 = G_t(\sigma_{t-1})^2,$$

where $G_t = (\frac{t}{t-1})^2$ for $t > 1$ ($G_1 = 1$). That is, it is assumed that the true variance in the pen increases by coefficient $(\frac{t}{t-1})^2$ (Kristensen et al., 2012).

It should be noted that the conjugate prior distribution of $(\sigma_t)^2$ is an inverse-gamma distribution (Gelman, Carlin, Stern, Dunson, Vehtari, and Rubin, 2004, p50). Hence, when the piglets are inserted into the pen ($t = 0$), the initial (prior) distribution of the variance is

$$\theta_0 \sim \text{Inv-Gamma} \left(\mathfrak{c}_0 = \frac{d-1}{2}, \mathfrak{d}_0 = \frac{(d-1)s_0^2}{2} \right), \quad (7)$$

with shape \mathfrak{c}_0 and scale \mathfrak{d}_0 where s_0^2 is the initial estimated sample variance of the live weight with sample size d . Given the nGSSM and the initial distribution (7), the estimate of the variance can now be updated when a new observation s_t^2 is obtained from the pen by using Theorem 3 and Corollary 1 in Appendix B.

4.3 Embedding the SSMs into the HMDP

The two SSMs described in the previous sections provide information about the mean weight and growth (μ_t, g_t) and the standard deviation σ_t of the weights in the pen. To embed this information into the HMDP these values have to be discretized (Nielsen et al., 2011).

Let $\mathbb{U}_{x_n} = \{\Pi_1, \dots, \Pi_{|\mathbb{U}_{x_n}|}\}$ be a set of disjoint intervals representing the partitioning of possible values of the continuous state variable x at stage n (e.g. $x = \hat{\mu}_n$). Moreover, given interval Π , let *centre point* π denote the centre of the interval. The set of possible values of the state variables in the HMDP related to information about weight is $\mathbb{W}_n = \mathbb{U}_{\hat{\mu}_n} \times \mathbb{U}_{\hat{\sigma}_n}$ and hence a state \mathbb{w}_n corresponds to area $\Pi_{\hat{\mu}_n} \times \Pi_{\hat{\sigma}_n}$ and is represented using the centre point $\mathbb{w}_n = (\pi_{\hat{\mu}_n}, \pi_{\hat{\sigma}_n})$. Similarly the corresponding set of possible values of the state variable related to information about growth is $\mathbb{G}_n = \mathbb{U}_{\hat{g}_n}$.

4.3.1 Transition probabilities

It is now possible to compute the transition probability $\Pr(\mathbb{w}_{n+1}, \mathbb{g}_{n+1} \mid \mathbb{w}_n, \mathbb{g}_n)$ used in (1) and (2). Since the mean and variance estimations are treated separately in different SSMs, this tran-

sition probability can be split into two parts:

$$\Pr(\mathbb{w}_{n+1}, \mathbb{g}_{n+1} \mid \mathbb{w}_n, \mathbb{g}_n) = \Pr(m_{n+1} = (\hat{\mu}_{n+1}, \hat{g}_{n+1}) \in \Pi_{\hat{\mu}_{n+1}} \times \Pi_{\hat{g}_{n+1}} \mid m_n = (\pi_{\hat{\mu}_n}, \pi_{\hat{g}_n})) \\ \cdot \Pr(m'_{n+1} = \hat{\sigma}_{n+1} \in \Pi_{\hat{\sigma}_{n+1}} \mid m'_n = \pi_{\hat{\sigma}_n}).$$

The first part can be calculated using the GSSM as

$$\Pr(m_{n+1} \in \Pi_{\hat{\mu}_{n+1}} \times \Pi_{\hat{g}_{n+1}} \mid m_n) = \int_{\Pi_{\hat{\mu}_{n+1}}} \int_{\Pi_{\hat{g}_{n+1}}} f_{(m_{n+1}|m_n)}(x, y) dy dx,$$

where the distribution of $(m_{n+1} \mid m_n)$ can be found using Theorem 2 in Appendix B. The second part can be calculated using the nGSSM as

$$\Pr(m'_{n+1} \in \Pi_{\hat{\sigma}_{n+1}} \mid m'_n) = \int_{\Pi_{\hat{\sigma}_{n+1}}} f_{(m'_{n+1}|m'_n)}(x) dx,$$

where the distribution of $(m'_{n+1} \mid m'_n)$ can be found using Theorem 4 in Appendix B.

4.3.2 Expected rewards

The expected reward given stage n and state $(q_n, \mathbb{w}_n, \mathbb{g}_n) = (q_n, (\pi_{\hat{\mu}_n}, \pi_{\hat{\sigma}_n}), \pi_{\hat{g}_n})$ in process p^2 can be calculated as the expected value of (3) minus the expected value of (4). The expected revenue of (3) can be written as

$$\sum_{k=q_n-q+1}^{q_n} \mathbb{E}(\tilde{w}_{(k)} \cdot p(\tilde{w}_{(k)}, \check{w}_{(k)})), \quad (8)$$

where $\tilde{w}_{(k)}$ and $\check{w}_{(k)}$ denote the carcass weight and leanness of the j th pig (based on the order statistics $w_{(k)}$, see Section 3.4). The carcass weight, $\tilde{w}_{(k)}$, of the j th pig is a fraction of live weight (Andersen, Pedersen, and Ogannisian, 1999):

$$\tilde{w}_{(k)} = 0.84w_{(k)} - 5.89 + \varepsilon, \quad (9)$$

where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ is a random error. Furthermore, Kristensen et al. (2012) proposed a rule of thumb for use in production units, which is used to compute the lean meat percentage $\check{w}_{(k)}$ at marketing:

$$\check{w}_{(k)} = \frac{-30(\bar{g}_{(k)} - \bar{g})}{4} + \bar{w},$$

where $\bar{g}_{(k)}$ denotes the average daily growth/gain of the k th pig until marketing, \bar{g} is the average daily growth in the herd, and \bar{w} is the average herd leanness percentage at marketing. The average daily growth t days after insertion into the pen is $\bar{g}_{(k)} = (w_{(k)} - \hat{\mu}_0)/t$, where $\hat{\mu}_0$ denotes the average weight at time of insertion into the pen.

The expected cost of (4) is:

$$c_f \sum_{k=1}^{q_n-q} \mathbb{E}(z_{(k)}). \quad (10)$$

and from (6), the ordered random variable $z_{(k)}$ equals:

$$z_{(k)} = k_{1t}w_{(k)} + k_{2t}g_{(k)}.$$

Note that the evaluation of (8) and (10) is rather complex since it involves calculating the mean of a piecewise reward function and the truncated normal distribution. However, the values of (8) and (10) can be simulated using a simple sorting procedure and given the fact that $w \sim N(\pi_{\hat{\mu}}, \pi_{\hat{\sigma}})$ where w denotes the weight of a pig randomly selected in the pen at the current stage.

5 Numerical example

To illustrate the functionality of the proposed optimization model, the HMDP is applied on three pens with different properties (average weekly gain). The average weekly gain of Pen 2 is assumed to be “normal” (an initial growth of 5.8 kg/week using Feed-mix 1), and Pens 1 and 3 grow twenty percent slower and faster, respectively, than Pen 2. Moreover, to initialize the three pens with the same conditions, the pigs are fed by the same feed-mix (Feed-mix 1) at the time of insertion into the pen.

5.1 Model parameters and observation data

To obtain time series of observations (\bar{w}_t, \bar{z}_t) and s_t^2 used by the GSSM and nGSSM a simulation model was developed. The model is based on the biological growth formulas in Jørgensen (1993). The simulation model and the generated data are available online for reproducibility (see Pourmoayed and Nielsen (2014b)).

An example is given in Figure 2 that shows the observed values of average live weight \bar{w}_t , average feed intake \bar{z}_t , and the standard deviation s_t (resulted from the simulation). It also gives the estimated information of live weight and growth (calculated using Bayesian updating with the GSSM and nGSSM) in the three pens. These values together with the other state variables are used to identify the current state in the HMDP. Note that the simulation is started with Feed-mix 1 and each time the feed-mix is changed, we continue the simulation using the new feed-mix.

The parameters used for the HMDP are given in Table 1. The parameter values have been obtained using information about finisher pig production (Danish conditions) and related literature (see the footnotes in Table 1).

Table 2 contains parameter values related to the GSSM and nGSSM. The values have been estimated with time series generated using the simulation model. More specifically, we used the expectation-maximization algorithm (Dethlefsen, 2001) to find V and W , and the initial posterior parameters $m_{0,1}$ and $C_{0,1}$ were estimated using the weight data at the time of inserting the piglets into the pen. For the nGSSM, the initial sample variance s_0^2 was calculated using the time series data and $d = 35$ is used as the number of weight estimates per day. Finally, note that each feed-mix implies a special growth rate in the pen ($\hat{g}_{0,f}$) and that feed-mixes with higher growth rates are more expensive in comparison with other feed-mixes (c_f).

To calculate the revenue of marketing in (3), the unit price function $p(\tilde{w}_{(j)}, \check{w}_{(j)})$ should be specified, which under Danish conditions is the sum of two piecewise linear functions $\tilde{p}(\tilde{w})$

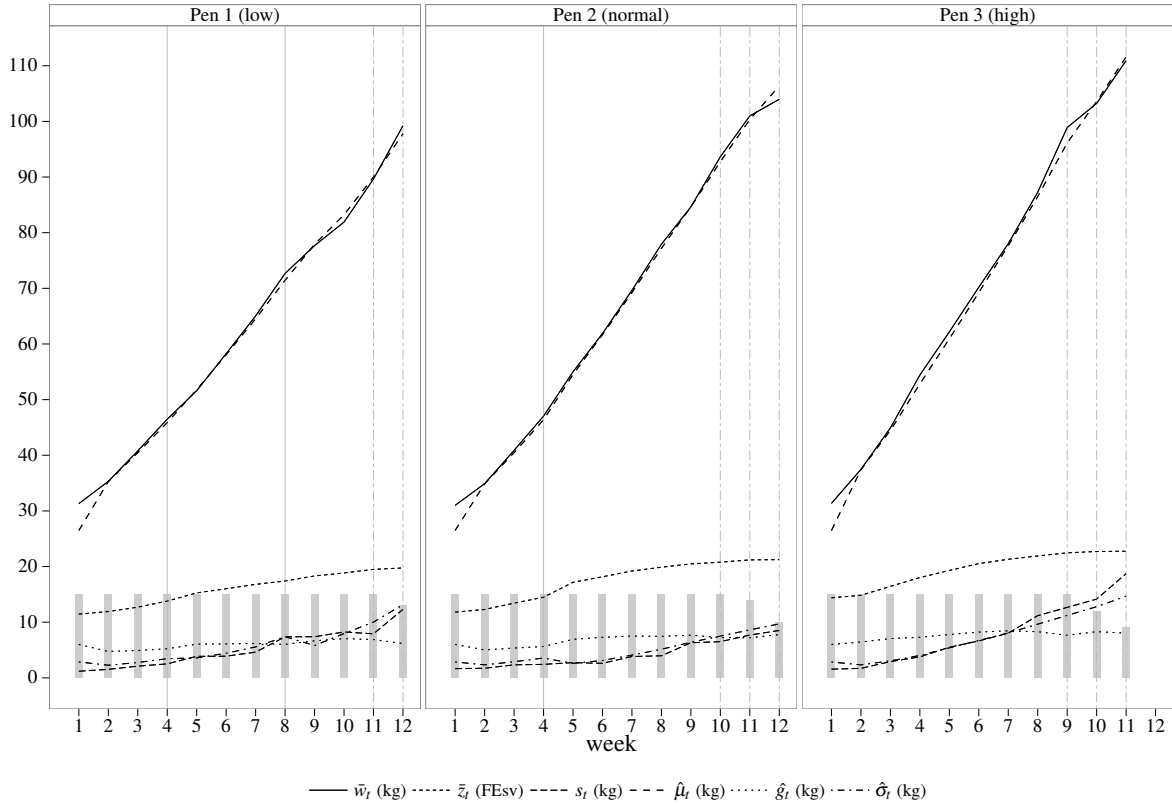


Figure 2: Observed and estimated information of live weight and growth rate in the three pens. Observed information are average live weight \bar{w}_t , average feed intake \bar{z}_t and standard deviation of live weight s_t per week (resulted from simulation). Estimated information are estimated means of live weight and growth rate, $\hat{\mu}_t$ and \hat{g}_t (computed using the GSSM), and estimated standard deviation of live weight $\hat{\sigma}_t$ (computed using the nGSSM). Bars show the number of pigs q_n in the pen before the optimal action is carried out. The vertical dotted and solid lines show the times when the marketing and feeding decisions occur in the system based on the optimal policy, respectively.

and $\check{p}(\check{w})$ related to the price of carcass and a bonus for the leanness percentage per kg meat, respectively. We define $\check{p}(\check{w})$ and $\check{p}(\check{w})$ based on the meat prices used in Kristensen et al. (2012)¹ as

¹For current prices see <http://www.danishcrown.dk/Ejer/Noteringer/Aktuel-svinenotering.aspx>

Table 1: Parameter values (HMDP).

Parameter	Value	Description
q^{\max}	15	Number of pigs inserted into the pen. ^a
b^{\max}	4	Maximum number of feeding phases. ^a
$ \mathbb{F} $	3	Number of available feed-mixes. ^a
t_f^{\min}	3	Minimum number of weeks using feed-mix f . ^a
t^{\max}	12	Maximum number of weeks in a growing period. ^a
t^{\min}	9	First possible week of marketing decisions. ^a
c_{newMix}	0	Cost of changing the feed-mix (DKK). ^a
c_{pig}	375	Cost of a piglet (DKK). ^{b,c}
c_f	{ 1.8, 1.88, 1.96 }	Unit cost of feed-mix $f = 1, \dots, 3$ (DKK/FEsv). ^{ad}
\bar{g}	6	Average weekly gain (kg) in the herd. ^c
\tilde{w}	61	Average leanness percentage in the herd. ^c
σ_ε	1.4	Standard deviation of ε . ^c

^a Value based on discussions with experts in Danish pig production. ^b Time series of Danish prices can be seen at <http://www.notering.dk/WebFrontend/>. ^c Value taken from Kristensen et al. (2012). ^d FEsv is the energy unit used for feeding the pigs in Denmark. One FEsv is equivalent to 7.72 MJ.

$$\tilde{p}(\tilde{w}) = \begin{cases} 0 & \tilde{w} < 50 \\ 0.2\tilde{w} - 2.7 & 50 \leq \tilde{w} < 60 \\ 0.1\tilde{w} + 3.3 & 60 \leq \tilde{w} < 70 \\ 10.3 & 70 \leq \tilde{w} < 86 \\ -0.1\tilde{w} + 18.9 & 86 \leq \tilde{w} < 95 \\ 9.3 & 95 \leq \tilde{w} < 100 \\ 9.1 & \tilde{w} \geq 100, \end{cases} \quad (11)$$

and

$$\check{p}(\check{w}) = 0.1(\check{w} - 61).$$

A plot of the two functions is given in Figure 3.

Finally, in order to initialize the HMDP, possible values of the state variables should be determined for each stage. For the discrete state variables (q_n, t_n, f_n) , the possible values are set according to the set of states defined in Section 3.2. Moreover, based on the discretization method in the beginning of Section 4.3, the continuous state variables $(\hat{\mu}_n, \hat{\sigma}_n, \hat{g}_n)$ are divided into the 11, 7 and 5 intervals, respectively. The centre points of these intervals are specified such that they represent the possible values of the weight and growth information in the system. An overview over the values of each state variable is given in Table 3.

Table 2: Parameter values (GSSM and nGSSM).

Parameter	Value	Explanation
<u>GSSM</u>		
V	$\begin{pmatrix} 0.066 & 0.027 \\ 0.027 & 0.012 \end{pmatrix}$	Observation variance. ^a
W	$\begin{pmatrix} 2.1 & -0.124 \\ -0.124 & 0.112 \end{pmatrix}$	System variance. ^a
$m_{0,1}$	$\begin{pmatrix} 26.49 \\ 5.8 \end{pmatrix}$	Initial prior mean weight and growth $m_{0,1} = (\hat{\mu}_0, \hat{g}_{0,1})$ for Feed-mix 1. ^a
$C_{0,1}$	$\begin{pmatrix} 4.26 & 0.32 \\ 0.32 & 0.53 \end{pmatrix}$	Initial prior covariance matrix for Feed-mix 1. ^a
k_{1i}	$\{0.134 - 0.004i + 0.0001i^2 : i = 1, \dots, 12\}$	Energy requirements (FEsv) per kg live weight. ^b
k_2	1.549	Energy requirement (FEsv) per kg gain. ^c
$\hat{g}_{0,f}$	$\{5.8, 6.3, 6.8\}$	Initial growth rate estimate of feed-mix $f = 1, \dots, 3$. ^d
<u>nGSSM</u>		
s_0^2	7.65	Initial sample variance (kg ²). ^a
d	35	Sample size (observations per day). ^e

^a Estimated based on time series generated using the simulation model. ^b Based on a linear approximation of the relation between feed intake and growth stated in Jørgensen (1993). ^c Value taken from Jørgensen (1993). ^d Value based on discussions with experts in Danish pig production. ^e Value used in the simulation model.

Table 3: Possible values of the state variables and the range of the centre points in the HMDP.

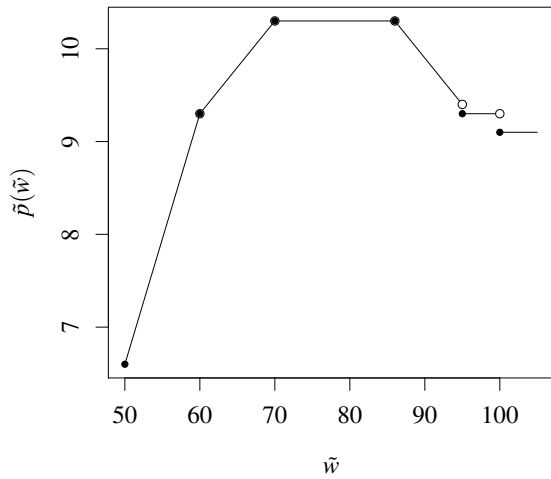
State / Week (n)	1	2	3	4	5	6	7	8	9	10	11	12
q_n	15	15	15	15	15	15	15	15	1-15	1-15	1-15	1-15
t_n (week)	1	1	1	1, 4	1, 4-5	1, 4-6	1, 4-7	1, 4-8	1, 4-9	1, 4-10	1, 4-11	1, 4-11
f_n	1	1	1	1-3	1-3	1-3	1-3	1-3	1-3	1-3	1-3	1-3
$\pi_{\hat{\mu}_n}$ (kg) ^a	7-47	14-54	20-61	28-68	35-75	42-82	49-88	56-96	63-103	70-110	77-116	84-124
$\pi_{\hat{\sigma}_n}$ (kg) ^a	1.6-6.4	2.1-6.9	2.6-7.4	3.1-7.9	3.6-8.4	4.1-8.9	4.6-9.4	5.1-9.9	5.6-10.4	6.1-10.9	6.6-11.4	7.1-11.9
$\pi_{\hat{g}_n}$ (kg) ^a	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2	4.4-8.2

^a Variables $\hat{\mu}_n$, $\hat{\sigma}_n$, \hat{g}_n are discretized into 11, 7, and 5 intervals, respectively. Rows $\pi_{\hat{\mu}_n}$, $\pi_{\hat{\sigma}_n}$, and $\pi_{\hat{g}_n}$ show the range of the possible values of $\hat{\mu}_n$, $\hat{\sigma}_n$ and \hat{g}_n .

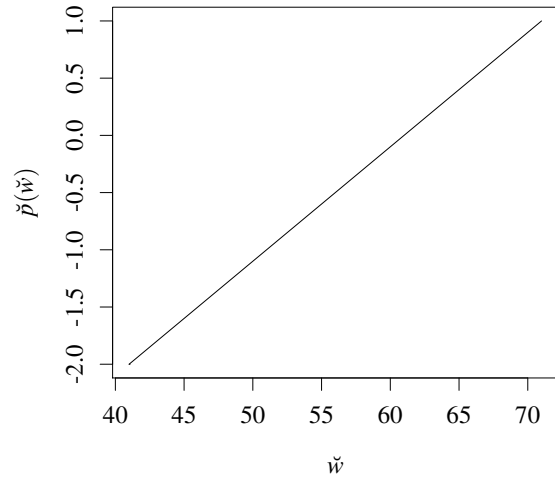
5.2 Model results

The HMDP was coded using C++ (gcc compiler) and R (R Core Team, 2015). The source code is available online (Pourmoayed and Nielsen, 2014b). After the model was built, the optimal policy was calculated with the modified policy iteration algorithm using the R package ‘‘MDP’’ (Nielsen, 2009). The resulting model consists of 802581 states and 5050446 actions (one stage of the founder process including states and actions of sub-processes).

The CPU time for building and solving the model was 268 and 94 seconds, respectively (Fujitsu laptop with i7-4800MQ CPU and 32 GB of memory running on a Windows 7 64 bit OS). Note that the model has only to be resolved when the model parameters change, e.g. a new estimation of V and W which might be re-estimated monthly. Therefore, a fast solution time is



(a) Carcass weight.



(b) Leanness.

Figure 3: Price functions (DKK/kg) given carcass weight ($\tilde{p}(\tilde{w}) = 0$ for $\tilde{w} < 50$) and leanness percentage.

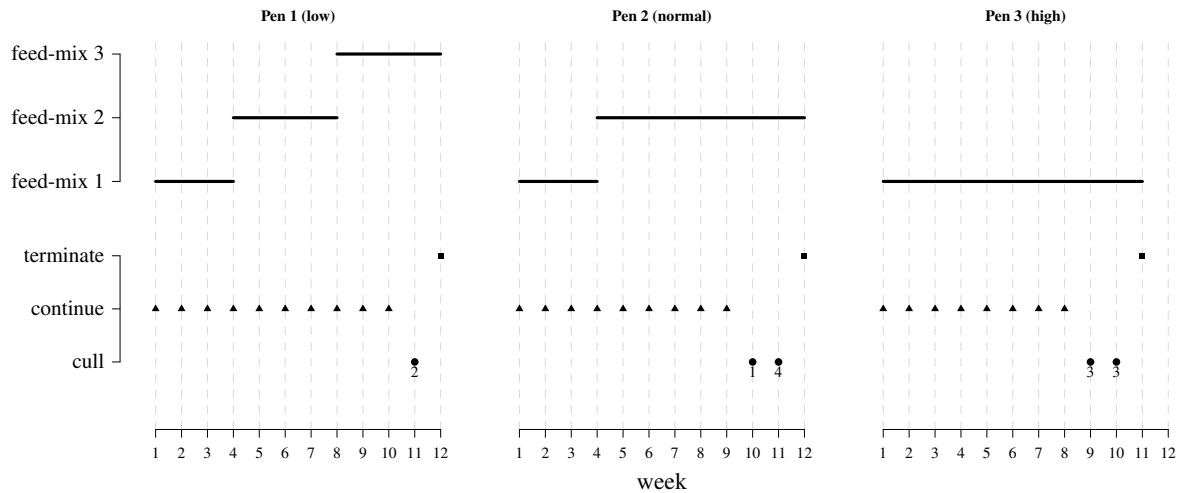


Figure 4: The optimal feeding and marketing decisions for the three pens. Upper part of each plot illustrates the optimal feed-mix (solid line) and the lower part shows the optimal marketing decision. Numbers close to cull actions (a_q) are the number of pigs culled.

not the primary focus.

The information from each pen, i.e. the values $q_n, t_n, f_n, \hat{\mu}_n, \hat{\sigma}_n$ and \hat{g}_n , is used to find the relevant state in the HMDP. Next, the optimal action is found using the calculated optimal policy. The resulting optimal feeding and marketing decisions (i.e. the sample path of the MDP) are illustrated in Figures 2 and 4 for each pen. In Figure 2 the vertical dotted and solid lines show weeks where marketing and feeding decisions are taken in the system. The bars show the number of pigs left in the pen before a (possible) marketing decision. For instance, in week 9, three pigs in Pen 3 are marketed. A detailed overview of the optimal decisions is given in Figure 4. Here the plot of each pen is separated into two parts. The solid line in the upper part shows the optimal feed-mix. A jump indicates that the optimal decision is to change the current feed-mix. In the lower part of each plot the optimal marketing decision is illustrated by means of symbols. For instance, the black dots indicate a culling action.

A closer look at Figure 4 shows that all pens start with Feed-mix 1. After 3 weeks, the feed-mix in Pen 1 (with the lowest weekly gain) changes to Feed-mix 2, resulting in a better growth rate compared to Feed-mix 1. Pen 1 (low growth) uses this feed-mix until week 8 and after that Feed-mix 3 is chosen for the remaining weeks because a higher growth is obtained (compared to using Feed-mix 2), and hence the appropriate live weight is reached at the end of the growing period. In Pen 2 (normal growth), we change the feed-mix in week 4 from Feed-mix 1 to Feed-mix 2 and until week 12 this feed-mix is used in the pen. In this pen, the average growth rate is appropriate and there is no need to use a more expensive feed-mix (Feed-mix 3) with a faster expected growth rate. Finally, in Pen 3 (high growth), the feed-mix remains unchanged since the pigs genetically grow fast in this pen using the cheapest feed-mix (Feed-mix 1) and they will have an appropriate live weight in the last weeks of the growing period.

The length of the growing period, i.e. the number of weeks before terminating the pen, differs between pens. Pens 1 and 2 are terminated at the maximum growth length (week 12) since a good slaughter weight is reached for the remaining pigs. However, Pen 3 is terminated in week 11. Due to the high growth rate in this pen, the average weight in week 11 is appropriate and the pen is terminated such that a new batch of piglets can be inserted into the pen earlier (new production cycle).

Individual marketing decisions are made in all pens to select the heaviest pigs for marketing. Usually pigs grow with different growth values in the pen and hence in the last weeks of the growing period (from week 9 to 12) they obtain different live weights. Hence, these decisions are made in order to market the pigs that are in the best slaughter weight interval (with a live weight approximately between 89 and 109 kg due to (9) and (11)). For instance, in Pen 2, the 4 heaviest pigs are culled in week 11. As a result, these individual marketing decisions lead to a decrease in the inhomogeneity between the remaining pigs in the pen, which implies a more consistent growth among the remaining pigs.

Changing the parameters of the model influences on the optimal policy. To make a small sensitivity analysis, three groups of scenarios are considered and compared with the basic scenario based on the parameters in Tables 1 and 2. In the first group of scenarios the starting time of the marketing period is changed by considering different values of t^{\min} under a fixed growing period ($t^{\max} = 12$). In the second group, the maximum length of the marketing period is altered by changing t^{\max} under a fixed starting time for marketing decisions ($t^{\min} = 9$) and in the third

Table 4: Three groups of scenarios to show the impact of changing model parameters. The basic scenario is based on the parameters in Tables 1 and 2 where $t^{\min} = 9$ and $t^{\max} = 12$ weeks.

Scenario group	Parameter change	Gross margin per week (DKK)
Basic	none	71.349
Group 1 - starting time of marketing period	$t^{\min} = 8$	71.355
	$t^{\min} = 10$	71.249
	$t^{\min} = 11$	70.428
Group 2 - maximum length of marketing period	$t^{\max} = 11$	34.155
	$t^{\max} = 13$	92.618
	$t^{\max} = 14$	104.644
	$t^{\max} = 15$	110.884
Group 3 - feed-mix unit cost	10% increase	27.444
	10% decrease	116.064

group different feed-mix unit costs are taken into account. Under each scenario, the optimal policy of the HMDP and the gross margin per week are calculated for comparison.

The results are shown in Table 4. In Group 1, a change in the starting time of possible marketing decisions (t^{\min}) has a small impact on the gross margin while in Group 2 the maximum length of the marketing period (t^{\max}) has a much higher impact on the gross margin per week. Therefore, it is better to increase the marketing length by extending the growing period t^{\max} compared to lowering t^{\min} . This illustrates the importance of the length of the growing period in the pen. Finally, in Group 3, a decrease/increase in the feed-mix unit cost gives a higher/lower gross margin. The effect is relatively high which shows that the profit of the production unit is extremely dependent on the feeding costs.

6 Conclusions and further research

In the production of growing pigs, the decision maker must consider feeding and marketing decisions simultaneously. In this paper, we presented a three-level HMDP which considers both feeding and marketing decisions at pen level.

We used a Bayesian approach to update the state of the system such that it contains the updated information based on previous measurements. More specifically, a GSSM is used to forecast mean weight and growth information based on online measurements and an nGSSM is used to forecast the weight variance within the pen. By embedding the SSMs into the HMDP, the model takes into account new online measurements. Both SSMs are embedded into the HMDP using a general discretization method.

A numerical example shows that the optimal policy adapts to different pen conditions (we used three pens with different genetic properties) and chooses actions which maximize the expected reward per time unit. Furthermore, a marginal sensitivity analysis illustrated the importance of the length of the growing period and feed-mix cost.

The model presented in this paper can be used as a part of a decision support system with online data such that the system state can be found using Bayesian updating and the optimal policy of the HMDP can determine the best feeding and marketing decisions at pen level. For simplicity we have assumed that the three alternative feed mixes available for the pigs are the same throughout the production period. In practice it would be natural to adjust the feed mixes to the various growth phases so that the alternatives taken into account depend on the age of the pigs. It would be straightforward to implement such a more realistic setup so it is not a limitation for a practical use. There are, however, some other limitations of the model that require more thorough consideration.

First, the model considers feeding and marketing decisions at pen level and ignores possible constraints at section or herd level. For instance, limitations in the feeding management system and the transportation strategy to the abattoir are currently ignored. That is, we assume weekly deliveries to the abattoir in the marketing period based on a cooperative agreement where culled pigs from each pen are grouped in one delivery. Hence, the transportation cost is fixed and can be ignored. This is the situation in many Danish herds since the majority of farmers in Denmark use a single abattoir which also handles the transport. To handle constraints and decisions about transportation costs (e.g. truck capacity), we need to extend the model from pen level to section or herd level. Given the current modeling framework, this extension may be difficult due to the curse of dimensionality since the number of states will grow dramatically. As a result there is a need for an approximation method to approximate the value function of the HMDP and find the best marketing policy in the herd. This can be done by using an approximate dynamic programming approach (Powell, 2007) and is a possible direction of future research.

Second, we may have weekly variations in the carcass price (11) and piglet cost in practice. This fact may have an influence on the marketing decisions but has been ignored in this study and in previous papers using HMDP models (Nielsen and Kristensen, 2015). Considering price variations in a model with marketing and feeding decisions is difficult since state variables related to price information have to be introduced into the model which will result in an exponential increase in the number of state variables. Two directions are possible. Either approximate dynamic programming methods are applied or other state variables are excluded from the model. Price variations can be analyzed using an SSM and Bayesian updating and embedded into an HMDP which is a subject of future research.

Finally, the model may be extended to handle information and decisions about diseases such as diarrhea and tail biting.

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A Notation

Since the paper uses techniques from both statistical forecasting and operations research, we had to make some choices with respect to notation. In general, we use capital letters for matrices and let A' denote the transpose of A . Capital blackboard bold letters are used for sets (e.g. \mathbb{W}_n and \mathbb{F}). Subscript indices indicate e.g. stage, week, and feed-mix and are separated using a comma. Superscript is only used to indicate the level in the HMDP except when lower and upper limits on ranges (e.g. t_f^{\min} and t^{\max}) are considered. Greek letters are used for some stochastic variables and their mean and variance such as θ_t and μ_t . Finally, accent \hat{x} (hat) is used to denote an estimate of x and accent \bar{x} (bar) the average of a group of x -variables. A description of the notation introduced in Section 3 and Section 4 is given in Tables 5 and 6, respectively.

Table 5: Notation - HMDP model (Section 3). Given in approx. the order introduced.

Symbol	Description
\mathbb{I}_n	Set of states at stage n .
$\mathbb{A}_n(i)$	Set of actions given stage n and state i .
$r_n(i, a)$	Reward at stage n given state i and action a .
$u_n(i, a)$	Expected length until the next decision epoch at stage n given state i and action a .
$\Pr(j n, i, a)$	Transition probability from state i at stage n to state j at the next stage under action a .
$\Pr_0(i)$	Initial probability of state i .
\mathfrak{p}^l	A process at level l (superscript is used to indicate level).
\mathcal{N}^l	Time horizon of process \mathfrak{p}^l at level l .
n^l, i^l, a^l	A stage, state, and action in process \mathfrak{p}^l .
q^{\max}	Number of pigs inserted into the pen.
b^{\max}	Maximum number of feeding phases.
t^{\max}	Latest week of pen termination.
t_f^{\min}	Minimum number of weeks for using feed-mix f .
t^{\min}	First possible week of marketing decisions.
f_n	Previous feed-mix used at stage/phase $n - 1$, $f_n \in \mathbb{F}$ (set of possible feed-mixes).
t_n	Starting time of phase/stage n (week number), $1 \leq t_n \leq t^{\max} - 1$.
q_n	Remaining pigs in the pen at stage n , $1 \leq q_n \leq q^{\max}$.
\mathbb{W}_n	Model information related to the weight of the pigs, $w_n \in \mathbb{W}_n$ (set of possible weight information).
\mathbb{G}_n	Model information related to the growth of the pigs, $g_n \in \mathbb{G}_n$ (set of possible growth information).
a_f	Child jump action for choosing feed-mix $f \in \mathbb{F}$.
a_{newMix}	Parent jump action related to changing the current feed-mix.
a_{term}	Parent jump action related to terminating a pen.
a_{cont}	Action related to continuing the production process without any marketing.
a_q	Action related to marketing the q heaviest pigs in the pen ($1 \leq q < q_n$).
c_{pig}	Unit cost of a piglet (DKK).
c_{newMix}	Fixed cost of changing the feed-mix (DKK).
c_f	Unit cost of feed-mix f (DKK/FEsv).
$w^{(k)}$	Weight of the k th pig in the pen (kg).
$z^{(k)}$	Weekly feed intake of the k th pig in the pen (FEsv).
$\tilde{w}^{(k)}$	Carcass weight of the k th pig in the pen (kg).
$\check{w}^{(k)}$	Lean meat percentage of the k th pig in the pen (%).
$\tilde{p}(\tilde{w})$	Unit price of carcass meat (DKK/kg).
$\check{p}(\check{w})$	Leanness bonus for 1 kg meat (DKK/kg).
$p(\tilde{w}, \check{w})$	Unit price of meat, $p(\tilde{w}, \check{w}) = \tilde{p}(\tilde{w}) + \check{p}(\check{w})$ (DKK/kg).

Table 6: Notation - GSSM and nGSSM models (Section 4). Given in approx. the order introduced.

Symbol	Description
<u>GSSM (Section 4.1)</u>	
μ_t	Mean weight in the pen at time t .
g_t	Mean growth in the pen at time t .
\bar{w}_t	Average weight estimate at time t , $\bar{w}_t = \sum_{k=1}^d \hat{w}_k / d$ where \hat{w}_k denotes the k th weight estimate and d is the number of weight observations per time unit (sample size).
\bar{z}_t	Average feed intake per pig at time t .
θ_t	Latent/unobservable variable(s).
y_t	Observable variable(s).
G	Design matrix of system equation.
F_t	Design matrix of observation equation.
ω_t	System noise, $\omega_t \sim N(0, W)$ where W denotes the system covariance matrix.
v_t	Observation error, $v_t \sim N(0, V)$ where V denotes the observation covariance matrix.
(m_0, C_0)	Mean and covariance matrix of prior given feed-mix f , $\theta_0 \sim N(m_0, C_0)$.
\mathbb{D}_t	Set of information available up to time t in the system, $\mathbb{D}_t = (y_1, \dots, y_t, m_0, C_0)$.
(m_t, C_t)	Mean and covariance matrix of posterior at time t , $(\theta_t \mathbb{D}_t) \sim N(m_t, C_t)$.
k_{1t}	Energy requirement per kg live weight (FEsv/kg).
k_2	Energy requirement per kg gain (FEsv/kg).
<u>nGSSM (Section 4.2)</u>	
σ_t	Standard deviation of the weights in the pen at time t .
s_t^2	Sample variance of weights in the pen at time t , $s_t^2 = \sum_{j=1}^d (\hat{w}_j - \bar{w}_t)^2 / (d - 1)$.
η_t	Natural parameter of the exponential family distribution.
$(\alpha_t, \mathfrak{b}_t)$	Shape and scale parameter of the Gamma distribution, $s_t^2 \sim \text{Gamma}(\alpha_t, \mathfrak{b}_t)$.
$(\mathfrak{c}_t, \mathfrak{d}_t)$	Shape and scale parameter of the inverse-Gamma distribution, $(\sigma_0)^2 \sim \text{Inv-Gamma}(\mathfrak{c}_t, \mathfrak{d}_t)$.
<u>Embedding into the HMDP (Section 4.3)</u>	
\mathbb{U}_{x_n}	Set of disjoint intervals representing the partitioning of possible values of estimate x at stage n , $\mathbb{U}_{x_n} = \{\Pi_1, \dots, \Pi_{ \mathbb{U}_{x_n} }\}$ where Π_k denotes interval k .
π_k	Centre point of Π_k .
ε	Random error in estimation of carcass weight given live weight, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ where σ_ε denotes the standard deviation.
\bar{g}	Average weekly gain (kg) in the herd.
$\bar{g}^{(k)}$	Average daily growth/gain of the k th pig until marketing.
\bar{w}	Average leanness percentage in the herd.
$\hat{\mu}_0$	Initial average weight (kg) at insertion time into the pen.

B Statistical models for Bayesian updating

Gaussian state space model (GSSM)

A GSSM includes a set of observable and latent/unobservable continuous variables. The set of latent variables $\theta_{\{t=0,1,\dots\}}$ evolves over time using *system equation* (written using matrix notation)

$$\theta_t = G_t \theta_{t-1} + \omega_t,$$

where $\omega_t \sim N(0, W_t)$ is a random term and G_t is a matrix of known values. We assume that the prior $\theta_0 \sim N(m_0, C_0)$ is given. Moreover, we have a set of observable variables $y_{\{t=1,2,\dots\}}$ (the data acquired from the online sensors) which are dependent on the latent variable using *observation equation*

$$y_t = F_t' \theta_t + v_t,$$

with $v_t \sim N(0, V_t)$. Here F is the design matrix of system equations with known values and F' denotes the transpose to matrix F .

The error sequences ω_t and v_t are internally and mutually independent. Hence given θ_t we have that y_t is independent of all other observations and in general the past and the future are independent given the present.

Let $\mathbb{D}_{t-1} = (y_1, \dots, y_{t-1}, m_0, C_0)$ denote the information available up to time $t-1$. Given the posterior of the latent variable at time $t-1$, we can use Bayesian updating (the Kalman filter) to update the distributions at time t (West and Harrison, 1997, Theorem 4.1).

Theorem 1 *Suppose that at time $t-1$ we have*

$$(\theta_{t-1} \mid \mathbb{D}_{t-1}) \sim N(m_{t-1}, C_{t-1}), \quad (\text{posterior at time } t-1).$$

then

$$\begin{aligned} (\theta_t \mid \mathbb{D}_{t-1}) &\sim N(b_t, R_t), & (\text{prior at time } t) \\ (y_t \mid \mathbb{D}_{t-1}) &\sim N(f_t, Q_t), & (\text{one-step forecast at time } t-1) \\ (\theta_t \mid \mathbb{D}_t) &\sim N(m_t, C_t), & (\text{posterior at time } t) \end{aligned}$$

where

$$\begin{aligned} b_t &= G_t m_{t-1}, & R_t &= G_t C_{t-1} G_t' + W_t \\ f_t &= F_t' b_t, & Q_t &= F_t' R_t F_t + V_t \\ e_t &= y_t - f_t, & B_t &= R_t F_t Q_t^{-1} \\ m_t &= b_t + B_t e_t, & C_t &= R_t - B_t Q_t B_t'. \end{aligned}$$

Note that the one-step forecast mean f_t only depends on m_{t-1} , i.e. we only need to keep the most recent conditional mean of θ_{t-1} to forecast the next value. Hence when making a prediction based on \mathbb{D}_{t-1} , we need only to store m_{t-1} . Similarly, the variance Q_t only depends on the

number of observations made, i.e. we can calculate a sequence Q_1, \dots, Q_t without knowing the observations y_1, \dots, y_t .

The distribution of $(m_{t+1} | m_t)$ can also be found (Nielsen et al., 2011, page 303).

Theorem 2 *The conditional random variable $(m_{t+1} | m_t)$ follows a multivariate normal distribution*

$$(m_{t+1} | m_t) \sim N(G_{t+1}m_t, B_{t+1}Q_{t+1}B'_{t+1}).$$

Non-Gaussian state space model (nGSSM)

An nGSSM relaxes the Gaussian assumption of the observed values, i.e. observations are not conditional Gaussian given the values of the latent variable θ_t . Instead the probability distribution of the observable variable y_t belongs to the exponential family, i.e. the density function is:

$$f(y_t | \eta_t, u_t) = \exp\left(\frac{x(y_t)\eta_t - a(\eta_t)}{u_t}\right) q(y_t, u_t), \quad (12)$$

with *natural parameter* η_t and *scale parameter* u_t . Functions $a(\eta_t)$, $x(y_t)$, and $q(y_t, u_t)$ are assumed known. The equation

$$g(\eta_t) = F_t \theta_t, \quad (13)$$

defines the impact of the latent variable θ_t on the natural parameter η_t . Here, $g(\eta_t)$ is a known function. Finally, to specify the full nGSSM model, a *system equation* has to be specified:

$$\theta_t = G_t \theta_{t-1} + \kappa_t,$$

with $\kappa_t \sim [0, H_t]$, meaning that κ_t has zero mean and a covariance matrix H_t . There is no assumption about a normal distribution. In other words, the distribution is only partially specified through its mean and variance (we use the notation $\kappa_t \sim [m_t, H_t]$).

As for the GSSM, the purpose of Bayesian updating is to estimate the latent variable θ_t using previous information $\mathbb{D}_{t-1} = (y_1, \dots, y_{t-1}, m_0, C_0)$ available up to time $t-1$. However, due to (13) we also estimate the parameter η_t . An updating procedure was presented by Kristensen et al. (2010, Section 8.5.4). Since there is no normality assumption, only an approximate analysis can be conducted. Moreover, the conjugate family of η_t must be known.

In our application a gamma distribution with shape parameter α_t and scale parameter b_t is used, i.e. $\eta_t = -1/\alpha_t b_t$, $V_t = 1/\alpha_t$, $a(\eta_t) = \ln(\frac{-1}{\eta_t})$, $x(y_t) = y_t$ and $q(y_t, \alpha_t) = y_t^{\alpha_t-1} \alpha_t^{\alpha_t} / \Gamma(\alpha_t)$ and the density becomes

$$f(y_t | \alpha_t, b_t) = \frac{\exp(-y_t/b_t) y_t^{\alpha_t-1}}{b_t^{\alpha_t} \Gamma(\alpha_t)}. \quad (14)$$

Moreover, the conjugate prior of $g(\eta_t)$ is an *Inverse-Gamma* distribution. As a result, the updating procedure (Kristensen et al., 2010, Section 8.5.4) reduces to the theorem below.

Theorem 3 *Suppose that at time $t-1$ we have*

$$(\theta_{t-1} | \mathbb{D}_{t-1}) \sim [m_{t-1}, C_{t-1}] \quad (\text{posterior at time } t-1),$$

Moreover, assume that $g(\eta_t) \sim \text{Inv-Gamma}(\mathbf{c}_t, \mathbf{d}_t)$, $g(\eta_t) = -1/\eta_t$, and that the density $f(y_t | \mathbf{a}_t, \mathbf{b}_t)$ equals (14). Then

$$\begin{aligned}(\boldsymbol{\theta}_t | \mathbb{D}_{t-1}) &\sim [b_t, R_t] \quad (\text{prior at time } t), \\(g(\eta_t) | \mathbb{D}_{t-1}) &\sim [f_t, Q_t] \quad (\text{prior of } g(\eta_t) \text{ at time } t), \\(\boldsymbol{\theta}_t | \mathbb{D}_t) &\sim [m_t, C_t] \quad (\text{posterior at time } t),\end{aligned}$$

where

$$\begin{aligned}b_t &= G_t m_{t-1}, & R_t &= G_t C_{t-1} G_t' + H_t, \\f_t &= F_t' b_t, & Q_t &= F_t' R_t F_t, \\m_t &= b_t + R_t F_t (f_t^* - f_t) / Q_t, & C_t &= R_t - R_t F_t F_t' R_t (1 - Q_t^* / Q_t) / Q_t, \\f_t^* &= \frac{\alpha_t^*}{\beta_t^*}, & Q_t^* &= \frac{\alpha_t^{*2}}{(\beta_t^*)^2 (\beta_t^* - 1)} \\ \alpha_t^* &= \alpha_t + \mathbf{a}_t' y_t, & \beta_t^* &= \beta_t + \mathbf{a}_t, \\ \alpha_t &= \frac{f_t^3}{Q_t} + f_t, & \beta_t &= \frac{f_t^2}{Q_t} + 1.\end{aligned}$$

PROOF Consider the updating procedure by Kristensen et al. (2010, Section 8.5.4) which consists of seven steps. The first three steps are the same, but repeated below for readability.

a) Posterior information for $\boldsymbol{\theta}_{t-1}$ at time $t - 1$:

$$(\boldsymbol{\theta}_{t-1} | \mathbb{D}_{t-1}) \sim [m_{t-1}, C_{t-1}],$$

b) Prior for $\boldsymbol{\theta}_t$ at time t :

$$(\boldsymbol{\theta}_t | \mathbb{D}_{t-1}) \sim [b_t, R_t], \quad b_t = G_t m_{t-1}, \quad R_t = G_t C_{t-1} G_t' + H_t.$$

c) Prior for $g(\eta_t)$ at time t :

$$(g(\eta_t) | \mathbb{D}_{t-1}) \sim [f_t, Q_t], \quad f_t = F_t' b_t, \quad Q_t = F_t' R_t F_t.$$

d) Approximate full prior for η_t at time t : According to our assumptions we have that $(g(\eta_t) | \mathbb{D}_{t-1}) \sim \text{Inv-Gamma}(\mathbf{c}_t, \mathbf{d}_t)$ where \mathbf{c}_t and \mathbf{d}_t are the shape and scale parameters, $g(\eta_t) = -1/\eta_t$, and that the density of y_t is (14).

In this step we need to identify the conjugate prior of $\eta_t = -1/g(\eta_t)$ using the general form of the conjugate prior with two parameters α_t and β_t (Kristensen et al., 2010):

$$f(\eta_t | \mathbb{D}_{t-1}) = c(\alpha_t, \beta_t) \exp(\alpha_t \eta_t - \beta_t a(\eta_t)),$$

where $c(\alpha_t, \beta_t)$ is a known function and $a(\eta_t) = \ln(-1/\eta_t)$ as defined in (12). If we suppose $y = \eta_t$ and $x = g(\eta_t)$, i.e. $y = h(x) = \frac{-1}{x}$, then by applying the transformation rule, the density function of η_t is

$$\begin{aligned} f(\eta_t | \mathbb{D}_{t-1}) &= f_x(h^{-1}(y) | \mathbb{D}_{t-1}) \frac{\partial h^{-1}(y)}{\partial y} \\ &= \frac{\mathfrak{d}_t^{c_t}}{\Gamma(c_t)} (-1/\eta_t)^{-c_t-1} \exp\left(-\frac{\mathfrak{d}_t}{-1/\eta_t}\right) \frac{1}{\eta_t^2} \\ &= \frac{\mathfrak{d}_t^{c_t}}{\Gamma(c_t)} (-\eta_t)^{c_t-1} \exp(\mathfrak{d}_t \eta_t) \\ &= \frac{\mathfrak{d}_t^{c_t}}{\Gamma(c_t)} \exp(\mathfrak{d}_t \eta_t - (c_t - 1) \ln(\frac{-1}{\eta_t})). \end{aligned}$$

Hence the parameters α_t and β_t in the conjugate prior of η_t become:

$$\alpha_t = \mathfrak{d}_t, \quad \beta_t = c_t - 1. \quad (15)$$

Finally, we fit α_t and β_t such that

$$\begin{aligned} \mathbb{E}(g(\eta_t) | \mathbb{D}_{t-1}) &= \frac{\mathfrak{d}_t}{c_t - 1} = \frac{\alpha_t}{\beta_t} = f_t, \\ \text{Var}(g(\eta_t) | \mathbb{D}_{t-1}) &= \frac{\mathfrak{d}_t^2}{(c_t - 1)^2 (c_t - 2)} = \frac{\alpha_t^2}{(\beta_t)^2 (\beta_t - 1)} = Q_t, \end{aligned} \quad (16)$$

implying that

$$\alpha_t = \frac{f_t^3}{Q_t} + f_t, \quad \beta_t = \frac{f_t^2}{Q_t} + 1.$$

- e) One step forecast of y_t : In this step we need to find the forecast distribution $f(y_t | \mathbb{D}_{t-1})$. According to the concepts of the nGSSM models, the general form of this distribution with two parameters α_t and β_t is (Kristensen et al., 2010):

$$f(y_t | \mathbb{D}_{t-1}) = \frac{c(\alpha_t, \beta_t) q(y_t, \mathbf{u}_t)}{c(\alpha_t + \phi_t x(y_t), \beta_t + \phi_t)},$$

where $q(y_t, \mathbf{u}_t)$ and $x(y_t)$ have been defined in (12) and $\phi_t = \frac{1}{\mathbf{u}_t}$. Using the values of α_t and β_t found in Step d, the forecast distribution $f(y_t | \mathbb{D}_{t-1})$ equals

$$f(y_t | \mathbb{D}_{t-1}) = \frac{1}{B(\mathfrak{a}_t, \mathfrak{c}_t)} \cdot \frac{1}{\mathfrak{d}_t / \mathfrak{a}_t} \cdot \left(\frac{y_t - 0}{\mathfrak{d}_t / \mathfrak{a}_t}\right)^{\mathfrak{a}_t - 1} \cdot \left(1 + \frac{y_t - 0}{\mathfrak{d}_t / \mathfrak{a}_t}\right)^{-\mathfrak{a}_t - \mathfrak{c}_t},$$

where $B(\mathfrak{a}_t, \mathfrak{c}_t) = \frac{\Gamma(\mathfrak{a}_t) \Gamma(\mathfrak{c}_t)}{\Gamma(\mathfrak{a}_t + \mathfrak{c}_t)}$. That is a *generalized beta prime* distribution denoted by β' (Crooks, 2013, page 50) and hence

$$(y_t | \mathbb{D}_{t-1}) \sim \beta'(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5), \quad (17)$$

with parameters: $\psi_1 = 0$ (location); $\psi_2 = \mathfrak{d}_t / \mathfrak{a}_t$ (scale); $\psi_3 = \mathfrak{a}_t$ (first shape); $\psi_4 = \mathfrak{c}_t$ (second shape); and $\psi_5 = 1$ (Weibull power parameter).

f) Posterior distributions for η_t and $g(\eta_t)$ at time t : The general form of posterior density function of η_t is:

$$f(\eta_t | \mathbb{D}_t) = c(\alpha_t^*, \beta_t^*) \exp(\alpha_t^* \eta_t - \beta_t^* a(\eta_t)),$$

with

$$\alpha_t^* = \alpha_t + \phi_t y_t = \alpha_t + \mathbf{a}_t y_t, \quad \beta_t^* = \beta_t + \phi_t = \beta_t + \mathbf{a}_t. \quad (18)$$

The last relation follows from $\phi_t = \frac{1}{u_t} = \mathbf{a}_t$. If we suppose $x = \eta_t$ and $y = g(\eta_t)$ then, based on relation $g(\eta_t) = -1/\eta_t$, we have $y = h(x) = -\frac{1}{x}$ and using the transformation rule, the posterior distribution of $g(\eta_t)$ is:

$$\begin{aligned} f_{g(\eta_t)}(g(\eta_t) | \mathbb{D}_t) &= f_x(h^{-1}(y) | \mathbb{D}_t) \frac{\partial h^{-1}(y)}{\partial y} \\ &= \frac{(\alpha_t^*)^{\beta_t^*+1}}{\Gamma(\beta_t^*+1)} \exp\left(\frac{-\alpha_t^*}{g(\eta_t)} - \beta_t^* \ln(g(\eta_t))\right) \frac{1}{g(\eta_t)^2} \\ &= \frac{(\alpha_t^*)^{\beta_t^*+1}}{\Gamma(\beta_t^*+1)} (g(\eta_t))^{-(\beta_t^*+1)-1} \exp\left(-\frac{\alpha_t^*}{g(\eta_t)}\right). \end{aligned} \quad (19)$$

It follows from (19) that $(g(\eta_t) | \mathbb{D}_t) \sim \text{Inv-Gamma}(\mathbf{c}_t^*, \mathfrak{d}_t^*)$ with

$$\mathbf{c}_t^* = \beta_t^* + 1, \quad \mathfrak{d}_t^* = \alpha_t^*.$$

Next, we fit α_t^* and β_t^* such that

$$\begin{aligned} f_t^* &= \mathbb{E}(g(\eta_t) | \mathbb{D}_t) = \frac{\mathfrak{d}_t^*}{\mathbf{c}_t^* - 1} = \frac{\alpha_t^*}{\beta_t^*}, \\ Q_t^* &= \text{Var}(g(\eta_t) | \mathbb{D}_t) = \frac{(\mathfrak{d}_t^*)^2}{(\mathbf{c}_t^* - 1)^2(\mathbf{c}_t^* - 2)} = \frac{(\alpha_t^*)^2}{(\beta_t^*)^2(\beta_t^* - 1)}, \end{aligned}$$

g) Posterior of θ_t at t : The posterior parameters m_t and C_t are

$$m_t = b_t + R_t F_t (f_t^* - f_t) / Q_t, \quad C_t = R_t - R_t F_t F_t' R_t (1 - Q_t^* / Q_t) / Q_t.$$

Since the above steps calculate the values stated in Theorem 3, this finishes the proof.

Corollary 1 *Given Theorem 3 and $F_t = 1$ we have that*

$$\begin{aligned} f_t &= b_t, & Q_t &= R_t, \\ m_t &= f_t^*, & C_t &= Q_t^*. \end{aligned}$$

Finally, the probability distribution of $(m_{t+1} | m_t)$ can be found.

Theorem 4 Under Theorem 3 and Corollary 1 and assuming $H_t = 0$, the conditional random variable $(m_{t+1} | m_t)$ follows a generalized beta prime distribution. That is,

$$(m_{t+1} | m_t) \sim \beta'(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5),$$

with parameters: $\psi_1 = G_{t+1}m_t\beta_t^*/(\beta_t^* + \alpha_{t+1})$ (location), $\psi_2 = \psi_1$ (scale), $\psi_3 = \alpha_{t+1}$ (first shape), $\psi_4 = \beta_t^* + 1$ (second shape), and $\psi_5 = 1$ (Weibull power) where α_{t+1} is the shape parameter of the exponential family distribution of y_{t+1} .

PROOF Assume that Theorem 3 and Corollary 1 hold. Then

$$(m_{t+1} | m_t) = (f_{t+1}^* | m_t) = \left(\frac{\alpha_{t+1}^*}{\beta_{t+1}^*} | m_t \right) = A_{t+1} + B_{t+1}(y_{t+1} | \mathbb{D}_t),$$

since based on (18) we have that

$$\frac{\alpha_{t+1}^*}{\beta_{t+1}^*} = A_{t+1} + B_{t+1}y_{t+1},$$

where

$$A_{t+1} = \frac{\alpha_{t+1}}{\beta_{t+1} + \alpha_{t+1}}, \quad B_{t+1} = \frac{\alpha_{t+1}}{\beta_{t+1} + \alpha_{t+1}}.$$

From (16) and since $H_t = 0$, we have that

$$\beta_{t+1} = \frac{\alpha_{t+1}}{f_{t+1}} = \frac{f_{t+1}^2}{Q_{t+1}} + 1 = \frac{(G_{t+1}f_t^*)^2}{G_{t+1}Q_t^*G_{t+1}'} + 1 = \frac{f_t^{*2}}{Q_t^*} + 1 = \left(\frac{\alpha_t^*}{\beta_t^*} \right)^2 \frac{\alpha_t^{*2}}{\beta_t^{*2}(\beta_t^* - 1)} + 1 = \beta_t^*,$$

which implies that

$$\alpha_{t+1} = f_{t+1}\beta_{t+1} = f_{t+1}\beta_t^* = G_{t+1}m_t\beta_t^*.$$

As a result we can compute A_{t+1} and B_{t+1} as

$$A_{t+1} = \frac{G_{t+1}m_t\beta_t^*}{\beta_t^* + \alpha_{t+1}}, \quad B_{t+1} = \frac{\alpha_{t+1}}{\beta_t^* + \alpha_{t+1}},$$

which are two scalars given m_t (since parameters α_{t+1} and β_t^* are known values given t).

Recall from (17), we have that $(y_{t+1} | \mathbb{D}_t) \sim \beta'(\check{\psi}_1, \check{\psi}_2, \check{\psi}_3, \check{\psi}_4, \check{\psi}_5)$, with parameters: $\check{\psi}_1 = 0$ (location), $\check{\psi}_2 = \mathfrak{d}_{t+1}/\alpha_{t+1}$ (scale), $\check{\psi}_3 = \alpha_{t+1}$ (first shape), $\check{\psi}_4 = \mathfrak{c}_{t+1}$ (second shape), and $\check{\psi}_5 = 1$ (Weibull power parameter). Hence we have that

$$(m_{t+1} | m_t) \sim \beta'(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5),$$

with parameters: $\psi_1 = A_{t+1}$, $\psi_2 = B_{t+1}\mathfrak{d}_{t+1}/\alpha_{t+1}$, $\psi_3 = \alpha_{t+1}$, $\psi_4 = \mathfrak{c}_{t+1}$, and $\psi_5 = 1$. Here we have used the property that if

$$X \sim \beta'(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5),$$

then (based on the transformation rule)

$$a + bX \sim \beta'(a + \psi_1, b\psi_2, \psi_3, \psi_4, \psi_5).$$

Note that due to (15), we have that

$$\psi_2 = B_{t+1} \vartheta_{t+1} / \mathfrak{a}_{t+1} = \frac{\vartheta_{t+1}}{\beta_t^* + \mathfrak{a}_{t+1}} = \frac{\alpha_{t+1}}{\beta_t^* + \mathfrak{a}_{t+1}} = A_{t+1},$$

and

$$\psi_4 = \mathfrak{c}_{t+1} = \beta_{t+1} + 1 = \beta_t^* + 1,$$

which finishes the proof.